New Methods in Robust Control

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For the period March 1988 through August 1991
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This report describes advances in robust control in three areas: Optimal H_∞ control, singular values, and dynamic inversion. The H_∞ results are a thorough treatment of the theory as it has been developed over the last three years. The structured singular value section describes an application of the technique to represent inertia parametric variations in the Space Station. The dynamic inversion section addresses global stability of aircraft pitch-axis dynamics using a dynamic inversion control approach.

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INTRODUCTION

This document is the first draft of the final report for the program New Methods in Robust Control. The emphasis of this program was to develop mathematical theory to help control system designers faced with challenging control problems associated with advanced aerospace vehicles. Relevant applications include flight control systems for new Air Force fighter/bomber aircraft, the F-18 HARV research vehicle, the NASP vehicle, the next generation launch system (ALS or NLS), and the Space Station. A common set of features characterizing these problems are:

1) Operation of vehicles in extreme ranges of flight environment
2) Significant uncertainties in dynamic models (e.g. aerodynamics)
3) Wide range of parameter variation during flight (e.g. mass properties)
4) Performance driven system designs (small safety margins)
5) Stability of system is critical to avoid loss of vehicle and human life.

Robustness of a control system is defined to be its capability to provide adequate performance in the presence of uncertainty. The importance of robustness in aerospace systems is well appreciated by the people who have to fly them. Bill Dana, NASA Dryden Chief Test Pilot, recently gave a presentation to the NASP community concerning lessons learned in the X-15 Flight Test Program as they would apply to NASP. His two key messages were:

1) Make the vehicle and its controls very robust with significant performance margins
2) Make very small incremental steps in the flight envelope (~0.5 Mach) during
the flight test program.

The theory developed on our program is motivated mainly by the need for robust controls in real vehicles.

Consistent with the goal of this program, we present three topics out of those worked on during the three years of this contract. The three topics are:

1) H_∞-optimal Control Theory
2) Structured Singular Values
3) Dynamic Inversion Control

The first of these topics is presented in a paper written by John Doyle and Keith Glover. The H_∞ theory presented there is a culmination of the research in that area that has been going on for the last decade. The theory in its "final" form is elegant from a mathematical viewpoint, but the practical value of that research lies beyond the solution of the H_∞-optimization problem itself (which has no guaranteed robustness properties). H_∞-optimization is one of the primary ingredients of the structured singular value technique. The primary practical value of the H_∞ theory is that we can now perform structured singular value synthesis more readily.

The second topic is presented in a document written by Mike Elgersma on mapping the mass-properties variations on the Space Station into a perturbation structure for a structured singular value design. Elgersma's effort to construct the perturbation structure was paid for by Space Station contract funds -- our contract only contributed the extra funds required for him to document the construction in a presentable form. There are now several papers in the open literature on structured singular value theory but there are far too few showing how the perturbation structure is made. We included Elgersma's example here as one application where the full power of the mathematical theory can be applied to a real-world (real-space?) application.

The third topic is presented in a document prepared by Blaise Morton and Dale Enns on the subject of dynamic inversion for aircraft pitch-axis control. An abstract of this paper was prepared for the Washington University/AFOSR workshop in St. Louis, August 15 and 16, 1991. The main new feature of this work is the global stability results for complementary dynamics. This result is perhaps the first non-local control stability result that applies directly to the nonlinear models used in industry for modern aircraft design.

A state-space approach to $\mathcal{H}_\infty$ optimal control*

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Abstract

Simple state-space formulae are derived for all controllers solving a standard $\mathcal{H}_\infty$ problem: for a given number $\gamma \geq 0$, find all controllers such that the $\mathcal{H}_\infty$ norm of the closed-loop transfer function is $< \gamma$. Under these conditions, a parametrization of all controllers solving the problem is given as a linear fractional transformation (LFT) on a contractive, stable free parameter. The state dimension of the coefficient matrix for the LFT equals that of the plant, and has a separation structure reminiscent of classical LQG (i.e., $\mathcal{H}_2$) theory. Indeed, the whole development is very reminiscent of earlier $\mathcal{H}_2$ results, especially those of Willems (1971). This paper directly generalizes the results in Doyle, Glover, Khargonekar, and Francis, 1989, and Glover and Doyle, 1988. Some aspects of the optimal case ($\leq \gamma$) are considered.

1 Introduction

1.1 Overview

The $\mathcal{H}_\infty$ norm defined in the frequency-domain for a stable transfer matrix $G(s)$ is

$$\|G\|_\infty := \sup_\omega \overline{\sigma}[G(j\omega)] \quad (\ \overline{\sigma} := \text{maximum singular value}\ )$$

The problem of analysis and synthesis of control systems using this norm arises in a number of ways. We assume the reader either is familiar with the engineering motivation for these problems, or is interested in the results of this paper for some other reason. This paper considers particular $\mathcal{H}_\infty$ optimal control problems that are direct generalizations of those considered in Doyle, Glover, Khargonekar, and Francis (1989), and Glover and Doyle (1988), hereafter referred to as DGKF and GD, respectively.

The basic block diagram used in this paper is

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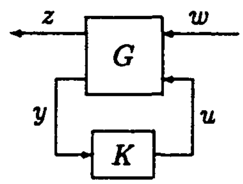


where G is the generalized plant and K is the controller. Only finite dimensional linear time-invariant (LTI) systems and controllers will be considered in this paper. The generalized plant G contains what is usually called the plant in a control problem plus all weighting functions. The signal w contains all external inputs, including disturbances, sensor noise, and commands, the output z is an error signal, y is the measured variables, and u is the control input. The diagram is also referred to as a linear fractional transformation (LFT) on K and G is called the coefficient matrix for the LFT. The resulting closed loop transfer function from w to z is denoted by $T_{zw} = \mathcal{F}_\ell(G, K)$.

The main $\mathcal{H}_\infty$ output feedback results of this paper as described in the abstract are presented in Section 4. The proofs of these results exploit the "separation" structure of the controller. If full information (x and w) is available, then the central controller is simply a gain matrix F_∞, obtained through finding a certain stable invariant subspace of a Hamiltonian matrix. Also, the optimal output estimator is an observer whose gain is obtained in a similar way from a dual Hamiltonian matrix. These special cases are described in Section 3. In the general output feedback case the controller can be interpreted as an optimal estimator for $F_\infty x$. Furthermore, the two Hamiltonians involved in this solution can be associated with full information and output estimation problems.

The proofs of these results are constructed out of a series of lemmas, several of which have some independent interest, particularly those involving state-space characterizations of mixed Hankel-Toeplitz operators. A possible contribution of this paper, beyond the new formulae and theorems, may be some of this technical machinery, most of which is developed in Section 2. The result is that the proofs of both the theorems and the lemmas leading to them are quite short. Furthermore, the development is reasonably self-contained, and the primary background required is a knowledge of elementary aspects of state-space theory, $\mathcal{L}_2$ spaces, and operators on $\mathcal{L}_2$, including projections and adjoints. More specialized knowledge about the connections between Riccati equations, spectral factorization, and Hamiltonian matrices would also be useful.

As mentioned, this paper is a direct generalization of DGKF, and contains a substantial repetition of material. Roughly speaking, we prove those results in GD which were stated without proof, using DGKF machinery, which considered a less general problem. An alternative approach in relaxing some of the assumptions in DGKF is to use loop-shifting techniques as in Zhou and Khargonekar (1988), GD, and more completely in Safonov *et al.* (1989). We also organize this paper much differently than DGKF. The results are presented in a conventional bottom-up linear order, with lemmas and theorems followed by their proofs, which in turn only use lemmas and theorems already proven. Readers interested in pursuing all the details of the proofs may find it more convenient than DGKF. This paper lacks the tutorial flavor of DGKF and the explicit connections with the more familiar $\mathcal{H}_2$ problem, although the $\mathcal{H}_2$ theory will be found lurking at every corner.

We also consider some aspects of generalizations to the $\leq$ case, primarily to indicate the problems encountered in the optimal case. A detailed derivation of the necessity the generalized conditions for the Full Information problem is given. In keeping with the style of GD and DGKF, we don't present a complete treatment of the $\leq$ case, but leave it for yet another day. Complete derivations of the optimal output feedback case can be found in Glover *et al.* (1989) using different techniques.

1.2 Historical perspective

This section is not intended as a review of the literature in $\mathcal{H}_\infty$ theory, nor even an attempt to outline the work that most closely touches on this paper. For a bibliography and review of the early $\mathcal{H}_\infty$ literature, the interested reader might see [Francis, 1987] and [Francis and Doyle, 1987], and an historical account of the results leading up to those in this paper may be found in DGKF. Instead, we will offer a slightly revisionist history, which lacks some factual accuracy, but has the advantage of more clearly emphasizing state-space methods and, more specifically, Willems' central role in $\mathcal{H}_\infty$ theory. This mildly fictionalized reconstruction tells things as they could have been, if only we'd been more clever, and thus contains a certain truth as valuable as that of a more factually accurate accounting. Besides, "historical perspectives" are often revisionist anyway, we're just admitting to it.

Zames' (1981) original formulation of $\mathcal{H}_\infty$ optimal control theory was in an input-output setting. Most solution techniques available at that time involved analytic functions (Nevanlinna-Pick interpolation) or operator-theoretic methods [Sarason, 1967; Adamjan *et al.*, 1978; Ball and Helton, 1983]. Indeed, $\mathcal{H}_\infty$ theory seemed to many to signal the beginning of the end for the state-space methods which had dominated control for the previous 20 years. Unfortunately, the standard frequency-domain approaches to $\mathcal{H}_\infty$ started running into significant obstacles in dealing with multi-input-output (MIMO) systems, both mathematically and computationally, much as the $\mathcal{H}_2$ theory of the 1950's had.

Not surprisingly, the first solution to a general rational MIMO $\mathcal{H}_\infty$ optimal control problem, presented in [Doyle, 1984], relied heavily on state-space methods, although more as a computational tool than in any essential way. The steps in this solution were as follows: parametrize all internally-stabilizing controllers via [Youla *et al.*, 1976]; obtain realizations of the closed-loop transfer matrix; convert the resulting model-matching problem into the equivalent 2×2-block general distance or best approximation problem involving mixed Hankel-Toeplitz operators; reduce to the Nehari problem (Hankel only); solve the Nehari problem by the procedure of Glover (1984). Both [Francis, 1987] and [Francis and Doyle, 1987] give expositions of this approach, which will be referred to as the "1984" approach.

In a mathematical sense, the 1984 procedure "solved" the $\mathcal{H}_\infty$ optimal control problem. Unfortunately, it involved a peculiar patchwork of techniques and the associated complexity of computation was substantial, involving several Riccati equations of increasing dimension, and formulae for the resulting controllers tended to be very complicated and have high state dimension. Nevertheless, much of the subsequent work in $\mathcal{H}_\infty$ control theory focused on the 2×2-block problems, either in the model-matching or general distance forms. This continued to provide a context for a stimulating interchange with operator theory, the benefits of which will hopefully continue to accrue. But from a control perspective, the $\mathcal{H}_\infty$ theory seemed once again to be headed into a cul-de-sac, but now with a Q in the corner. The solution has turned out to involve an even more radical emphasis on state-space theory.

In addition to providing controller formulae that are simple and expressed in terms of plant data, the methods in DGKF and this paper are a fundamental departure from the earlier work described above. In particular, the Youla parametrization and the resulting 2×2-block model-matching problem of the 1984 solution are avoided entirely; replaced by a more purely state-space approach involving observer-based compensators, a pair of 2×1 block problems, and a separation argument. The operator theory still plays a central role (as does Redheffer's work [Redheffer, 1960] on linear fractional transformations), but its use is more straightforward. The key to this was a return to simple and familiar state-space tools, in the style of Willems (1971), such as completing the square, and the connection between frequency domain inequalities (e.g. $\|G\|_\infty < 1$), Riccati equations, and spectral factorization. In essence, one only needed to think about how Willems would do it, and the rest is simply technical detail.

The state-space theory of $\mathcal{H}_\infty$ can be carried much further, by generalizing time-invariant to time-varying, infinite horizon to finite horizon, and finite dimensional to infinite dimensional. A flourish of activity has begun on these problems and the already numerous results indicate, not surprisingly, that many of the results of this paper generalize *mutatis mutandis*, to these cases. In fact, a cynic might express a sense of *déjà vu*, that despite all the rhetoric, $\mathcal{H}_\infty$ theory has come to look much like LQG, circa 1970 (or even more specifically, LQ differential games). A more charitable view might be that current $\mathcal{H}_\infty$ theory, rather than ending the reign of state-space, reaffirms the power of its computational machinery and the wisdom of its visionaries, exemplified by Jan Willems.

1.3 Notation

The notation is fairly standard. The Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ consist of square-integrable functions on the imaginary axis with analytic continuation into, respectively, the right and left half-plane. The Hardy space $\mathcal{H}_\infty$ consists of bounded functions with analytic continuation into the right half-plane. The Lebesgue spaces $\mathcal{L}_2 = \mathcal{L}_2(-\infty,\infty)$, $\mathcal{L}_{2+} = \mathcal{L}_2[0,\infty)$, and $\mathcal{L}_{2-} = \mathcal{L}_2(-\infty,0]$ consist, respectively of square-integrable functions on $(-\infty,\infty)$, $[0,\infty)$, and $(-\infty,0]$, and $\mathcal{L}_\infty$ consists of bounded functions on $(-\infty,\infty)$. As interpreted in this paper, $\mathcal{L}_\infty$ will consist of functions of frequency, $\mathcal{L}_{2+}$ and $\mathcal{L}_{2-}$ functions of time, and $\mathcal{L}_2$ will be used for both.

We will make liberal use of the Hilbert space isomorphism, via the Laplace transform and the Paley-Wiener theorem, of $\mathcal{L}_2 = \mathcal{L}_{2+} \oplus \mathcal{L}_{2-}$ in the time-domain with $\mathcal{L}_2 = \mathcal{H}_2 \oplus \mathcal{H}_2^\perp$ in the frequency-domain and of $\mathcal{L}_{2+}$ with $\mathcal{H}_2$ and $\mathcal{L}_{2-}$ with $\mathcal{H}_2^\perp$. In fact, we will normally not make any distinction between a time-domain signal and its transform. Thus we may write $w \in \mathcal{L}_{2+}$ and then treat w as if $w \in \mathcal{H}_2$. This style streamlines the development, as well as the notation, but when any possibility of confusion could arise, we will make it clear whether we are working in the time- or frequency- domain.

All matrices and vectors will be assumed to be complex. A transfer matrix in terms of state-space data is denoted

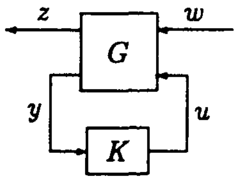
$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI - A)^{-1}B + D$$

For a matrix $M \in C^{p \times r}$, M' denotes its conjugate transpose, $\bar{\sigma}(M) = \rho(M'M)^{1/2}$ denotes its maximum singular value, $\rho(M)$ denotes its spectral radius (if $p = r$), and $M^\dagger$ denotes the Moore-Penrose pseudo-inverse of M. Im denotes image, ker denotes kernel, and $G^\sim(s) := G(-\bar{s})'$. For operators, Γ^* denotes the adjoint of Γ. The prefix $\mathcal{B}$ denotes the open unit ball and the prefix $\mathcal{R}_c$ denotes complex-rational.

The orthogonal projections P_+ and P_- map $\mathcal{L}_2$ to, respectively, $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ (or $\mathcal{L}_{2+}$ and $\mathcal{L}_{2-}$). For $G \in \mathcal{L}_\infty$, the Laurent or multiplication operator $M_G : \mathcal{L}_2 \to \mathcal{L}_2$ for frequency-domain $w \in \mathcal{L}_2$ is defined by $M_G w = Gw$. The norms on $\mathcal{L}_\infty$ and $\mathcal{L}_2$ in the frequency-domain were defined in Section 1.1. Note that both norms apply to matrix or vector-valued functions. The unsubscripted norm $\|\bullet\|$ will denote the standard Euclidean norm on vectors. We will omit all vector and matrix dimensions throughout, and assume that all quantities have compatible dimensions.

1.4 Problem statement

Consider the system described by the block diagram



Both G and K are complex-rational and proper, K is constrained to provide internal stability. We will denote the transfer functions from w to z as T_{zw} in general and for a feedback connection (LFT) as above we also write $T_{zw} = \mathcal{F}_\ell(G, K)$. This section discusses the assumptions on G that will be used. In our application we shall have state models of G and K. Then ***internal stability*** will mean that the states of G and K go to zero from all initial values when $w = 0$.

Since we will restrict our attention exclusively to proper, complex-rational controllers which are stabilizable and detectable, these properties will be assumed throughout. Thus the term controller will be taken to mean a controller which satisfies these properties. Controllers that have the additional property of being internally-stabilizing will be said to be *admissible.* Although we are taking everything to be complex, in the special case where the original data is real (e.g. G is real-rational) then all the of the results (such as K) will also be real.

The problem to be considered is to find all admissible $K(s)$ such that $\|T_{zw}\|_\infty < \gamma$ ($\leq \gamma$). The realization of the transfer matrix G is taken to be of the form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

compatible with the dimensions $z(t) \in \mathcal{C}^{p_1}$, $y(t) \in \mathcal{C}^{p_2}$, $w(t) \in \mathcal{C}^{m_1}$, $u(t) \in \mathcal{C}^{m_2}$, and the state $x(t) \in \mathcal{C}^n$. The following assumptions are made:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable

(A2) D_{12} is full column rank with $\begin{bmatrix} D_{12} & D_\perp \end{bmatrix}$ unitary and D_{21} is full row rank with $\begin{bmatrix} D_{21} \\ \tilde{D}_\perp \end{bmatrix}$ unitary.

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

Assumption (A1) is necessary for the existence of stabilizing controllers. The assumptions in (A2) mean that the penalty on $z = C_1 x + D_{12} u$ includes a nonsingular, normalized penalty on the control u, and that the exogenous signal w includes both plant disturbance and sensor noise, and the sensor noise weighting is normalized and nonsingular. Relaxation of (A2) leads to singular control problems.

Assumption (A3) relaxes the DGKF assumptions that (C_1, A) is detectable and $D_{12}' C_1 = 0$, and (A4) relaxes (A, B_1) stabilizable and $B_1 D_{21}' = 0$. Assumptions (A3) and (A4) are made for a technical reason: together with (A1) it guarantees that the two Hamiltonian matrices in the corresponding $\mathcal{H}_2$ problem belong to $dom(Ric)$. It is tempting to suggest that (A3) and (A4) can be dropped, but they are, in some sense, necessary for the methods in this paper to be applicable. A further discussion of the assumptions and their possible relaxation will be discussed in Section 5.2.

It can be assumed, without loss of generality, that $\gamma = 1$ since this is achieved by the scalings $\gamma^{-1} D_{11}$, $\gamma^{-1/2} B_1$, $\gamma^{-1/2} C_1$, $\gamma^{1/2} B_2$, $\gamma^{1/2} C_2$, and $\gamma^{-1} K$. This will be done implicitly for many of the proofs and statements of this paper.

2 Preliminaries

This section reviews some mathematical preliminaries, in particular the computation of the various norms of a transfer matrix G. Consider the transfer matrix

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.1}$$

with A stable (i.e., all eigenvalues in the left half-plane).

The norm $\|G\|_\infty$ arises in a number of ways. Suppose that we apply an input $w \in \mathcal{L}_2$ and consider the output $z \in \mathcal{L}_2$. Then a standard result is that $\|G\|_\infty$ is the induced norm of the multiplication operator M_G, as well as the Toeplitz operator $P_+ M_G : \mathcal{H}_2 \to \mathcal{H}_2$.

$$\|G\|_\infty = \sup_{w \in B\mathcal{L}_2} \|z\|_2 = \sup_{w \in B\mathcal{L}_{2+}} \|P_+ z\|_2 = \sup_{w \in B\mathcal{H}_2} \|P_+ M_G w\|_2$$

The rest of this section involves additional characterizations of the norms in terms of state-space descriptions. Section 2.1 collects some basic material on the Riccati equation and the Riccati operator which play an essential role in the development of both theories. Section 2.3 reviews some results on Hankel operators and introduces the 2×1-block mixed Hankel-Toeplitz operator result that will play a key role in the $\mathcal{H}_\infty$ FI problem.

Section 2.4 includes two lemmas on characterizing inner transfer functions and their role in certain LFT's and Section 2.5 considers the stabilizability and detectability of feedback systems.

2.1 The Riccati operator

Let A, Q, R be complex $n \times n$ matrices with Q and R Hermitian. Define the $2n \times 2n$ Hamiltonian matrix

$$H := \begin{bmatrix} A & R \\ Q & -A' \end{bmatrix}$$

If we begin by assuming H has no eigenvalues on the imaginary axis, then it must have n eigenvalues in Re $s < 0$ and n in Re $s > 0$. Consider the two n-dimensional spectral subspaces $\mathcal{X}_-(H)$ and $\mathcal{X}_+(H)$: the former is the invariant subspace corresponding to eigenvalues in Re $s < 0$; the latter, to eigenvalues in Re $s > 0$. Finding a basis for $\mathcal{X}_-(H)$, stacking the basis vectors up to form a matrix, and partitioning the matrix, we get

$$\mathcal{X}_-(H) = \mathrm{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{2.2}$$

where $X_1, X_2 \in C^{n \times n}$, and

$$H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} T_X, \quad \mathrm{Re}\ \lambda_i(T_X) < 0\ \forall\ i \tag{2.3}$$

If X_1 is nonsingular, or equivalently, if the two subspaces

$$\mathcal{X}_-(H), \quad \mathrm{Im} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{2.4}$$

are complementary, we can set $X := X_2 X_1^{-1}$. Then X is uniquely determined by H, i.e., $H \mapsto X$ is a function, which will be denoted Ric; thus, $X = Ric(H)$. We will take the domain of Ric, denoted $dom(Ric)$, to consist of Hamiltonian matrices H with two properties, namely, H has no eigenvalues on the imaginary axis and the two subspaces in (2.4) are complementary.

For ease of reference, these will be called the stability property and the complementarity property, respectively. The following well-known results give some properties of X as well as verifiable conditions under which H belongs to $dom(Ric)$. See, for example, Section 7.2 in [Francis, 1987], Theorem 12.2 in [Wonham, 1985], and [Kucera, 1972]

Lemma 2.1 *Suppose $H \in dom(Ric)$ and $X = Ric(H)$. Then*

(a) X is Hermitian

(b) X satisfies the algebraic Riccati equation

$$A'X + XA + XRX - Q = 0$$

(c) $A + RX$ is stable

Lemma 2.2 *Suppose H has no imaginary eigenvalues, R is either positive semi-definite or negative semi-definite, and (A, R) is stabilizable. Then $H \in dom(Ric)$.*

Lemma 2.3 *Suppose H has the form*

$$H = \begin{bmatrix} A & -BB' \\ -C'C & -A' \end{bmatrix}$$

with (A, B) stabilizable and $\operatorname{rank}\begin{bmatrix} A' + j\omega I & C' \end{bmatrix} = n \ \forall \omega$. Then $H \in dom(Ric)$, $X = Ric(H) \geq 0$, and $\ker(X) \subset \mathcal{X} :=$ stable unobservable subspace.

By stable unobservable subspace we mean the intersection of the stable invariant subspace of A with the unobservable subspace of (A, C). Note that if $(C, -A)$ is detectable, then $Ric(H) > 0$. Also, note that $\ker(X) \subset \mathcal{X} \subset \ker(C)$, so that the equation $XM = C'$ always has a solution for M, for example the least-squares solution given by $X^{\dagger}C'$.

We may extend the domain of Ric by relaxing the stability requirement. Even if H has eigenvalues on the imaginary axis, it must have at least n eigenvalues in Re $s \leq 0$. Suppose that we now choose some n-dimensional invariant subspace, again denoted by $\mathcal{X}_-(H)$, corresponding to n eigenvalues in Re $s \leq 0$ and a corresponding basis as in (2.2), but now satisfying

$$H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} T_X, \quad \text{Re } \lambda_i(T_X) \leq 0 \ \forall \ i \tag{2.5}$$

This subspace is not uniquely determined by H, but if it still satisfies the complementarity property, then we can set $X := X_2 X_1^{-1}$ as before, if this X is also Hermitian. We may thus define a new map $\overline{Ric}$, whose domain $dom(\overline{Ric})$ will be taken to consist of Hamiltonian matrices H with the property that an $\mathcal{X}_-(H)$ exists satisfying the complementarity condition and with the resulting $X := X_2 X_1^{-1}$ Hermitian. To show that this is actually a map, we have to verify that X is uniquely determined, which is not always the case. In fact, the conditions under which $\overline{Ric}$ is actually a map are intimately connected with the conditions on existence of $\mathcal{H}_\infty$ optimal controllers. It turns out that for the cases of interest in the present paper, whenever H is in $dom(\overline{Ric})$, the subspace will be uniquely determined. Thus whenever $\overline{Ric}$ is needed, it will be a well-defined map, but this must be proven. Fortunately, these cases can essentially be reduced to spectral factorization problems and standard theory can be applied (e.g. Gohberg, Lancaster and Rodman (1986)).

We may further extend the domain of $\overline{Ric}$ by relaxing the complementarity condition. The minimal requirement we will place on $\mathcal{X}_-(H)$ is that (2.5) hold and that

$$X_1' X_2 = X_2' X_1 \tag{2.6}$$

is Hermitian. Note that this condition also does not depend on the particular choice of basis taken in (2.2). It is convenient to define $dom(\overline{\overline{Ric}})$ to be the set of those H for which a subspace $\mathcal{X}_-(H)$ exists and satisfies (2.5) and (2.6). Once again, the map $\overline{\overline{Ric}}$, from $dom(\overline{\overline{Ric}})$ to n dimensional subspaces of C^{2n} (this is a Grassman manifold) does not always exist as the subspace is not uniquely determined by H.

The same remarks about $\overline{Ric}$ as a map apply here to $\overline{\overline{Ric}}$. These have been introduced in order to treat the optimal case, but their use will be limited as this case is not analysed in detail. Note that $dom(Ric) \subset dom(\overline{Ric}) \subset dom(\overline{\overline{Ric}})$. Also, if $H \in dom(Ric)$ then $\overline{Ric}$ and $\overline{\overline{Ric}}$ are obviously well-defined maps and $Ric(H) = \overline{Ric}(H)$.

2.2 Computing $\mathcal{H}_\infty$ norm

For the transfer matrix $G(s)$ in (2.1), with A stable, define the Hamiltonian matrix

$$H \;:=\; \begin{bmatrix} A + BR^{-1}D'C & BR^{-1}B' \\ -C'(I - DD')^{-1}C & -(A + BR^{-1}D'C)' \end{bmatrix} \tag{2.7}$$

$$= \begin{bmatrix} A & 0 \\ -C'C & -A' \end{bmatrix} + \begin{bmatrix} B \\ -C'D \end{bmatrix} R^{-1} \begin{bmatrix} D'C & B' \end{bmatrix} \tag{2.8}$$

where $R = I - D'D$. The following lemma is essentially from [Anderson, 1967], [Willems, 1971], and [Boyd *et al.*, 1989].

Lemma 2.4 *I. Let $\bar{\sigma}(D) < 1$, then the following conditions are equivalent:*

(a) $\|G\|_\infty < 1$

(b) H has no eigenvalues on the imaginary axis

(c) $H \in dom(Ric)$

(d) $H \in dom(Ric)$ and $Ric(H) \geq 0$ ($Ric(H) > 0$ if (C, A) is observable)

II. Let $\bar{\sigma}(D) < 1$, then the following conditions are equivalent:

(a) $\|G\|_\infty \leq 1$

(b) $H \in dom(\overline{Ric})$

(c) $H \in dom(\overline{Ric})$ *and* $\overline{Ric}(H)$ *is unique with* $\overline{Ric}(H) \geq 0$ *(*$\overline{Ric}(H) > 0$ *if* (C, A) *is observable)*

Proof From

$$(I - G^{\sim}G)(s) = \left[\begin{array}{cc|c} A & 0 & -B \\ -C'C & -A' & C'D \\ \hline D'C & B' & R \end{array}\right]$$

it is immediate that H is the A-matrix of $(I - G^{\sim}G)^{-1}$. It is easy to check using the PBH test that this realization has no uncontrollable or unobservable modes on the imaginary axis. Thus H has no eigenvalues on the imaginary axis iff $(I - G^{\sim}G)^{-1}$ has no poles there, i.e., $(I - G^{\sim}G)^{-1} \in \mathcal{RL}_\infty$. So to prove the equivalence of (Ia) and (Ib) it suffices to prove that

$$\|G\|_\infty < 1 \Leftrightarrow (I - G^{\sim}G)^{-1} \in \mathcal{RL}_\infty$$

If $\|G\|_\infty < 1$, then $I - G(j\omega)^*G(j\omega) > 0, \forall\omega$, and hence $(I - G^{\sim}G)^{-1} \in \mathcal{RL}_\infty$. Conversely, if $\|G\|_\infty \geq 1$, then $\bar{\sigma}[G(j\omega)] = 1$ for some ω, i.e., 1 is an eigenvalue of $G(j\omega)^*G(j\omega)$, so $I - G(j\omega)^*G(j\omega)$ is singular. Thus (Ia) and (Ib) are equivalent.

The equivalence of (Ib) and (Ic) follows from Lemma 2.2, and the equivalence of (Ic) and (Id) follows from Lemma 2.1 and standard results for solutions of Lyapunov equations.

The proof of part II is more involved and is given by the established results on spectral factorization as in Gohberg *et al.*(1986), since $I - G^{\sim}G \geq 0$ for all $s = j\omega$. ∎

In part II it was assumed that $\bar{\sigma}(D) < 1$ so that the Hamiltonian matrix could be defined. Alternatives that avoid this are to consider Linear Matrix Inequalities or the deflating subspaces of matrix pencils. This is discussed more in Section 5.2.5.

Lemma 2.4 suggests the following way to compute an $\mathcal{H}_\infty$ norm: select a positive number γ; test if $\|G\|_\infty < \gamma$ by calculating the eigenvalues of H; increase or decrease γ accordingly; repeat. Thus $\mathcal{H}_\infty$ norm computation requires a search, over either γ or ω. We should not be surprised by similar characteristics of the $\mathcal{H}_\infty$-optimal control problem. A somewhat analogous situation occurs for matrices with the norms $\|M\|_2^2 = \text{trace}(M^*M)$ and $\|M\|_\infty = \bar{\sigma}[M]$. In principle, $\|M\|_2^2$ can be computed exactly with a finite number of operations, as can the test for whether $\bar{\sigma}(M) < \gamma$ (e.g. $\gamma^2 I - M^*M > 0$), but the value of $\bar{\sigma}(M)$ cannot. To compute $\bar{\sigma}(M)$ we must use some type of iterative algorithm.

2.3 Mixed Hankel-Toeplitz Operators

It will be useful to characterize some additional induced norms of $G(s)$ in (2.1) and its associated differential equation

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \end{aligned} \tag{2.9}$$

with A stable. We will prove several lemmas that will be useful in the rest of the paper. It is convenient to describe all the results in the frequency-domain and give all the proofs in time-domain.

Consider first the problem of using an input $w \in \mathcal{L}_{2-}$ to maximize $\|P_+ z\|_2^2$. This is exactly the standard problem of computing the Hankel norm of G (i.e., the induced norm of the Hankel operator $P_+ M_G : \mathcal{H}_2^\perp \to \mathcal{H}_2$), and can be expressed in terms of the Gramians L_c and L_o

$$AL_c + L_c A' + BB' = 0 \qquad A'L_o + L_o A + C'C = 0 \tag{2.10}$$

Although this result is well-known, we will include a time-domain proof similar in technique to the proofs of the new results in this paper.

Lemma 2.5 $\displaystyle \sup_{w \in B\mathcal{L}_{2-}} \|P_+ z\|_2^2 = \sup_{w \in B\mathcal{H}_2^\perp} \|P_+ M_G w\|_2^2 = \rho(L_o L_c)$

Proof Assume (A, B) is controllable; otherwise, restrict attention to the controllable subspace. Then L_c is invertible and $w \in \mathcal{L}_{2-}$ can be used to produce any $x(0) = x_0$ given $x(-\infty) = 0$. The proof is in two steps. First,

$$\inf_{w \in \mathcal{L}_{2-}} \left\{ \|w\|_2^2 \;\middle|\; x(0) = x_0 \right\} = x_0' L_c^{-1} x_0 \tag{2.11}$$

To show this, we can differentiate $x(t)' L_c^{-1} x(t)$ along the solution of (2.9) for any given input w as follows:

$$\frac{d}{dt}(x' L_c^{-1} x) = \dot{x}' L_c^{-1} x + x' L_c^{-1} \dot{x} = x'(A' L_c^{-1} + L_c^{-1} A)x + w' B' L_c^{-1} x + x' L_c^{-1} B w$$

Use of (2.10) to substitute for $A' L_c^{-1} + L_c^{-1} A$ and completion of the squares give

$$\frac{d}{dt}(x' L_c^{-1} x) = \|w\|^2 - \|w - B' L_c^{-1} x\|^2$$

Integration from $t = -\infty$ to $t = 0$ with $x(-\infty) = 0$ and $x(0) = x_0$ gives

$$x_0' L_c^{-1} x_0 = \|w\|_2^2 - \|w - B' L_c^{-1} x\|_2^2 \le \|w\|_2^2$$

If $w(t) = B' e^{-A't} L_c^{-1} x_0 = B' L_c^{-1} e^{(A + BB' L_c^{-1})t} x_0$ on $(-\infty, 0]$, then $w = B' L_c^{-1} x$ and equality is achieved, thus proving (2.11).

Second, given $x(0) = x_0$ and $w = 0$, the norm of $z(t) = Ce^{At} x_0$ can be found from

$$\|P_+ z\|_2^2 = \int_0^\infty x_0' e^{A't} C' C e^{At} x_0 dt = x_0' L_o x_0$$

These two results can be combined as in Section 2 of [Glover, 1984]:

$$\sup_{w \in B\mathcal{L}_{2-}} \|P_+ z\|_2^2 = \sup_{0 \ne w \in \mathcal{L}_{2-}} \frac{\|P_+ z\|_2^2}{\|w\|_2^2} = \max_{x_0 \ne 0} \frac{x_0' L_o x_0}{x_0' L_c^{-1} x_0} = \rho(L_o L_c) \qquad \blacksquare$$

If $\|G\|_\infty < 1$ then by Lemmas 2.1 and 2.4, the Hamiltonian matrix H in (2.8) is in $dom(Ric)$, $X = Ric(H) \geq 0$, $A + BB'X$ is stable and

$$A'X + XA + C'C + (XB + C'D)R^{-1}(B'X + D'C) = 0 \tag{2.12}$$

Similarly, if $\bar{\sigma}(D) < 1$ and $\|G\|_\infty \leq 1$ then by Lemma 2.4, the Hamiltonian matrix H in (2.8) is in $dom(\overline{Ric})$, $X = \overline{Ric}(H) \geq 0$, $A + BB'X$ has eigenvalues in the closed left half plane and (2.12) holds.

The following lemma offers additional consequence of bounds on $\|G\|_\infty$. In fact, this simple time-domain characterization and its proof form the basis for the entire development to follow.

Lemma 2.6 *I. Suppose* $\|G\|_\infty < 1$ *and* $x(0) = x_0$. *Then*

$$\sup_{w \in \mathcal{L}_{2+}} (\|z\|_2^2 - \|w\|_2^2) = x_0' X x_0$$

and the sup *is achieved.*

II. Suppose that $\|G\|_\infty \leq 1$, $\bar{\sigma}(D) < 1$, *and* $x(0) = x_0$. *Then*

$$\sup_{w \in \mathcal{L}_{2+}} (\|z\|_2^2 - \|w\|_2^2) = x_0' X x_0$$

Proof: We can differentiate $x(t)'Xx(t)$ as above, use the Riccati equation (2.12) to substitute for $A'X + XA$, and complete the squares to get

$$\frac{d}{dt}(x'Xx) = -\|z\|^2 + \|w\|^2 - \|R^{-1/2}[Rw - (B'X + D'C)x]\|^2$$

If $w \in \mathcal{L}_{2+}$, then $x \in \mathcal{L}_{2+}$, so integrating from $t = 0$ to $t = \infty$ gives

$$\|z\|_2^2 - \|w\|_2^2 = x_0'Xx_0 - \|R^{-1/2}[Rw - (B'X + D'C)x]\|_2^2 \leq x_0'Xx_0 \tag{2.13}$$

For Part I, if we let $w = -R^{-1}(B'X + D'C)x = B'Xe^{[A+BR^{-1}(B'X+D'C)]t}x_0$, then $w \in \mathcal{L}_{2+}$ because $A + BR^{-1}(B'X + D'C)$ is stable. Thus the inequality in (2.13) can be made an equality and the proof is complete. Note that the sup is achieved for a w which is a linear function of the state.

For Part II, $A + BR^{-1}(B'X + D'C)$ may have imaginary axis eigenvalues, hence the inequality in (2.13) is still valid, but may not give the supremum. A sequence of functions w_ϵ can however be constructed to approach the supremum by considering $X_\epsilon = Ric(H_\epsilon)$ where

$$H_\epsilon = \begin{bmatrix} A & 0 \\ -C'C & -A' \end{bmatrix} + \begin{bmatrix} B \\ -C'D \end{bmatrix} (R + \epsilon^2 I)^{-1} \begin{bmatrix} D'C & B' \end{bmatrix}$$

Then for $w_\epsilon = (R + \epsilon^2 I)^{-1}(B'X_\epsilon + D'C)x$

$$\|z\|_2^2 - \|w_\epsilon\|_2^2 = x_0'X_\epsilon x_0 + \epsilon^2\|w_\epsilon\|_2^2 \leq x_0'Xx_0$$

Finally taking the limit as $\epsilon \to 0$ gives the result by uniqueness of $X = \lim_{\epsilon \to 0} X_\epsilon$. ∎

Now suppose that the input is partitioned so that $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, $D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$, $G(s) = \begin{bmatrix} G_1(s) & G_2(s) \end{bmatrix}$, and w is partitioned conformally. Then $\|G_2\|_\infty < 1$ iff

$$H_W := \begin{bmatrix} A & 0 \\ -C'C & -A' \end{bmatrix} + \begin{bmatrix} B_2 \\ -C'D_2 \end{bmatrix} R_2^{-1} \begin{bmatrix} D_2'C & B_2' \end{bmatrix}$$

is in $dom(Ric)$, where $R_2 := I - D_2'D_2$. Similarly, $\bar{\sigma}(D_2) < 1$ and $\|G\|_\infty \leq 1$ iff $H_W \in dom(\overline{Ric})$. In either case, define $W = \overline{Ric}(H_W)$, which will be unique, and let

$$w \in \mathcal{W} := \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \middle| w_1 \in \mathcal{H}_2^\perp, w_2 \in \mathcal{L}_2 \right\} \tag{2.14}$$

We are interested in a test for $\sup_{w \in \mathcal{BW}} \|P_+ z\|_2 < 1 \quad (\leq 1)$, or equivalently

$$\sup_{w \in \mathcal{BW}} \|\Gamma w\|_2 < 1 \quad (\leq 1) \tag{2.15}$$

where $\Gamma = P_+[M_{G_1} \; M_{G_2}] : \mathcal{W} \to \mathcal{H}_2$ is a mixed Hankel-Toeplitz operator:

$$\Gamma \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = P_+ \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad w_1 \in \mathcal{H}_2^\perp, \;\; w_2 \in \mathcal{L}_2$$

Note that Γ is the sum of the Hankel operator $P_+ M_G P_-$ with the Toeplitz operator $P_+ M_{G_2} P_+$. The following lemma generalizes Lemma 2.4 ($B_1 = 0, D_1 = 0$) and Lemma 2.5 ($B_2 = 0, D_2 = 0$).

Lemma 2.7 *I. (2.15) holds with $<$ iff the following two conditions hold:*

(i) $H_W \in dom(Ric)$

(ii) $\rho(WL_c) < 1$

II. (2.15) holds with $\leq$ iff the following two conditions hold:

(i) $H_W \in dom(\overline{Ric})$

(ii) $\rho(WL_c) \leq 1$

Proof As in Lemma 2.5, assume (A, B) is controllable; otherwise, restrict attention to the controllable subspace. By Lemma 2.4, condition (i) is necessary for (2.15) for both cases, so we will prove that given condition (i), (2.15) holds iff condition (ii) holds. By definition of $\mathcal{W}$, if $w \in \mathcal{W}$ then

$$\|P_+ z\|_2^2 - \|w\|_2^2 = \|P_+ z\|_2^2 - \|P_+ w_2\|_2^2 - \|P_- w\|_2^2$$

Note that the last term only contributes to $\|P_+ z\|_2^2$ through $x(0)$. Thus if L_c is invertible, then Lemma 2.6 and (2.11) yield

$$\sup_{w\in\mathcal{W}} \left\{ \|P_+z\|_2^2 - \|w\|_2^2 \;\middle|\; x(0)=x_0 \right\} \;=\; x_0'Wx_0 - x_0'L_c^{-1}x_0 \tag{2.16}$$

For part I we will prove the equivalent statement that $\rho(WL_c) \geq 1$ iff $\sup_{w\in\mathcal{BW}} \|\Gamma w\|_2 \geq 1$. The supremum is achieved in (2.16) for some $w \in \mathcal{W}$ that can be constructed from the previous lemmas. Since $\rho(WL_c) \geq 1$ iff $\exists\ x_0 \neq 0$ such that the right-hand side of (2.16) is ≥ 0, we have, by (2.16), that $\rho(WL_c) \geq 1$ iff $\exists\ w \in \mathcal{W}$, $w \neq 0$ such that $\|P_+z\|_2^2 \geq \|w\|_2^2$. But this is true iff $\sup_{w\in\mathcal{BW}} \|\Gamma w\|_2 \geq 1$.

For part II, note that (2.15) holds with $\leq$ iff

$$\sup_{w\in\mathcal{BW}} \|\Gamma w\|_2^2 - \|w\|_2^2 \leq 0$$

which by (2.16) is iff $\rho(WL_c) \leq 1$. ■

The FI proof of Section 3.3 will make use of the adjoint $\Gamma^* : \mathcal{H}_2 \to \mathcal{W}$, which is given by

$$\Gamma^* z = \begin{bmatrix} P_-(G_1^\sim z) \\ G_2^\sim z \end{bmatrix} = \begin{bmatrix} P_-G_1^\sim \\ G_2^\sim \end{bmatrix} z \tag{2.17}$$

where $P_-Gz := P_-(Gz) = (P_-M_G)z$. That the expression in (2.17) is actually the adjoint of Γ is easily verified from the definition of the inner product on vector-valued $\mathcal{L}_2$, expressed in the frequency-domain as

$$< x_1, x_2 > := \frac{1}{2\pi}\int_{-\infty}^{\infty} x_1(j\omega)^* x_2(j\omega)d\omega \tag{2.18}$$

The adjoint of $\Gamma : \mathcal{W} \to \mathcal{H}_2$ is the operator $\Gamma^* : \mathcal{H}_2 \to \mathcal{W}$ such that $< z, \Gamma w > = < \Gamma^* z, w >$ for all $w \in \mathcal{W}$, $z \in \mathcal{H}_2$. Directly using the definition in (2.18), we get

$$\begin{aligned} < z, \Gamma w > &= < z, P_+(G_1w_1 + G_2w_2) > = < z, G_1w_1 > + < z, G_2w_2 > \\ &= < P_-(G_1^\sim z), w_1 > + < G_2^\sim z, w_2 > \\ &= < \Gamma^* z, w > \end{aligned}$$

2.4 LFT's and inner matrices

A transfer function G in $\mathcal{RH}_\infty$, is called *inner* if $G^\sim G = I$, and hence $G(j\omega)^*G(j\omega) = I$ for all ω. Note that G inner implies that G has at least as many rows as columns. For G inner, and any $q \in \mathcal{C}^m$, $w \in \mathcal{L}_2$, then $\|G(j\omega)q\| = \|q\|$, $\forall\omega$, and $\|Gw\|_2 = \|w\|_2$. Because of these norm preserving properties inner matrices will be central to several of the proofs. In this section we give a characterization of inner functions and some properties of linear fractional transformations. First, we present a state-space characterization of inner transfer functions analogous to Lemma 2.4 that is well-known and simple to verify (see [Anderson, 1967], [Wonham, 1985], [Glover, 1984]).

Lemma 2.8 *Suppose* $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *with* (C, A) *detectable and* $L_o = L_o'$ *satisfies*

$$A'L_o + L_oA + C'C = 0.$$

Then

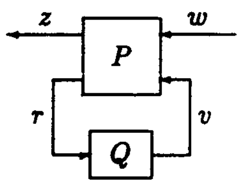
(a) $L_o \geq 0$ *iff* A *is stable*

(b) $D'C + B'L_o = 0$ *implies* $G^{\sim}G = D'D$

(c) $L_o \geq 0$, (A, B) *controllable, and* $G^{\sim}G = D'D$ *implies* $D'C + B'L_o = 0$.

The next lemma considers linear fractional transformations with inner matrices and is based on the work of Redheffer (1960).

Lemma 2.9 *Consider the following feedback system,*



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{RH}_\infty$$

Suppose that $P^{\sim}P = I$, $P_{21}^{-1} \in \mathcal{RH}_\infty$, *and* Q *is a proper rational matrix. Then the following are equivalent:*

(a) The system is internally stable and well-posed, and $\|T_{zw}\|_\infty < 1$.

(b) $Q \in \mathcal{RH}_\infty$ *and* $\|Q\|_\infty < 1$.

Proof (b) $\Rightarrow$ (a). Internal stability and well-posedness follow from $P, Q \in \mathcal{RH}_\infty$, $\|P_{22}\|_\infty \leq 1$, $\|Q\|_\infty < 1$, and a small gain argument. To show that $\|T_{zw}\|_\infty < 1$ consider the closed-loop system at any frequency $s = j\omega$ with the signals fixed as complex constant vectors. Let $\|Q\|_\infty =: \epsilon < 1$ and note that $T_{wr} = P_{21}^{-1}(I - P_{22}Q) \in \mathcal{RH}_\infty$. Also let $\kappa := \|T_{wr}\|_\infty$. Then $\|w\| \leq \kappa\|r\|$, and P inner implies that $\|z\|^2 + \|r\|^2 = \|w\|^2 + \|v\|^2$. Therefore,

$$\|z\|^2 \leq \|w\|^2 + (\epsilon^2 - 1)\|r\|^2 \leq [1 - (1 - \epsilon^2)\kappa^{-2}]\|w\|^2$$

which implies $\|T_{zw}\|_\infty < 1$.

(a) $\Rightarrow$ (b). To show that $\|Q\|_\infty < 1$ suppose there exist a (real or infinite) frequency ω and a constant nonzero vector r such that at $s = j\omega$, $\|Qr\| \geq \|r\|$. Then setting $w = P_{21}^{-1}(I - P_{22}Q)r$, $v = Qr$ gives $v = T_{vw}w$. But as above, P inner implies that $\|z\|^2 + \|r\|^2 = \|w\|^2 + \|v\|^2$ and hence $\|z\|^2 \geq \|w\|^2$, which is impossible since $\|T_{zw}\|_\infty < 1$. It follows that $\bar{\sigma}(Q(j\omega)) < 1$ for all ω, i.e., $\|Q\|_\infty < 1$, since Q is rational.

Finally, Q has a right-coprime factorization $Q = NM^{-1}$ with $N, M \in \mathcal{RH}_\infty$. We shall show that $M^{-1} \in \mathcal{RH}_\infty$. Since $T_{vw}P_{21}^{-1} = Q(I - P_{22}Q)^{-1}$ it has the right-coprime factorization $T_{vw}P_{21}^{-1} = N(M - P_{22}N)^{-1}$ But since $T_{vw}P_{21}^{-1} \in \mathcal{RH}_\infty$, so does $(M - P_{22}N)^{-1}$. This implies that the winding number of $\det(M - P_{22}N)$, as s traverses the Nyquist contour, equals zero. Furthermore, since $\det(M - \alpha P_{22}N) \neq 0$ for all α in [0,1] and all $s = j\omega$ (this uses the fact that $\|P_{22}\|_\infty \leq 1$ and $\|Q\|_\infty < 1$), we have that the winding number of $\det M$ equals zero too. Therefore, $Q \in \mathcal{RH}_\infty$ and the proof is complete. ∎

2.5 LFT's and stability

In this section, we consider the stabilizability and detectability of feedback systems. The proofs in this section are very routine and use standard techniques, principally the PBH test for controllability or observability, so they will only be sketched.

Recall the realization of G from Section 1.4 and suppose that $A \in \mathcal{C}^{n \times n}$, and that z, y, w and u have dimension p_1, p_2, m_1, and m_2, respectively. Thus $C_1 \in \mathcal{C}^{p_1 \times n}$, $B_2 \in \mathcal{C}^{n \times m_2}$, and so on. Now suppose we apply a controller K with stabilizable and detectable realization to G to obtain T_{zw}. For the following lemma, we do not need the assumptions from Section 1.4 on G for the output feedback problem.

Lemma 2.10 *The feedback connection of the realizations for G and K is,*

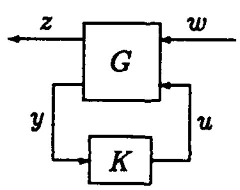
(a) detectable if rank $\begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ *for all* $Re\lambda \geq 0$.

(b) stabilizable if rank $\begin{bmatrix} A - \lambda I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ *for all* $Re\lambda \geq 0$.

Proof Form the closed-loop state-space matrices and perform a PBH test for controllability and observability. It is easily checked that any unobservable or uncontrollable modes must occur at λ violating the above rank conditions (see Limebeer and Halikias (1988) or Glover(1989) for more details), hence giving the results. ■

3 Full Information and Full Control Problems

In this section we discuss four problems from which the output feedback solutions will be constructed via a separation argument. These special problems are central to the whole approach taken in this paper, and as we shall see, they are also important in their own right. All pertain to the standard block diagram,



but with different structures for G. The problems are labeled

FI. Full information

FC. Full control

DF. Disturbance feedforward (to be considered in section 4.1)

OE. Output estimation (to be considered in section 4.1)

FC and OE are natural duals of FI and DF, respectively. The DF solution can be easily obtained from the FI solution, as shown in Section 4.1. The output feedback solutions will be constructed out of the FI and OE results. A dual derivation could use the FC and DF results.

The FI and FC problems are not, strictly speaking, special cases of the output feedback problem, as they do not satisfy all of the assumptions. Each of the four problems inherits certain of the assumptions A1–A4 from Section 1.4 as appropriate. The terminology and assumptions will be discussed in the subsections for each problem. In each of the four cases, the results are necessary and sufficient conditions for the existence of a controller such that $\|T_{zw}\|_\infty < \gamma$ and the family of all controllers such that $\|T_{zw}\|_\infty < \gamma$. In all cases, K must be admissible.

The $\mathcal{H}_\infty$ solution involves two Hamiltonian matrices, H_∞ and J_∞ which are defined as follows:

$$R := D_{1\bullet}'D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D_{1\bullet} := [D_{11} \;\; D_{12}]$$

$$\tilde{R} := D_{\bullet 1}D_{\bullet 1}' - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D_{\bullet 1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$$

$$H_\infty := \begin{bmatrix} A & 0 \\ -C_1'C_1 & -A' \end{bmatrix} - \begin{bmatrix} B \\ -C_1'D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}'C_1 & B' \end{bmatrix} \tag{3.1}$$

$$J_\infty := \begin{bmatrix} A' & 0 \\ -B_1B_1' & -A \end{bmatrix} - \begin{bmatrix} C' \\ -B_1D_{\bullet 1}' \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1}B_1' & C \end{bmatrix} \tag{3.2}$$

If $H_\infty \in dom(\overline{Ric})$ then let X_1, X_2 be any matrices such that

$$H_\infty \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} T_X, \quad X_1'X_2 = X_2'X_1, \quad \text{Re } \lambda_i(T_X) \le 0 \;\forall\, i \tag{3.3}$$

Similarly if $J_\infty \in dom(\overline{Ric})$ then let Y_1, Y_2 be any matrices such that

$$J_\infty \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} T_Y, \quad Y_1'Y_2 = Y_2'Y_1 \quad \text{Re } \lambda_i(T_Y) \le 0 \;\forall\, i \tag{3.4}$$

Further if in addition $H_\infty \in dom(Ric)$ and/or $J_\infty \in dom(Ric)$ then define,

$$X_\infty := X_2X_1^{-1}, \quad Y_\infty := Y_2Y_1^{-1} \tag{3.5}$$

Finally define the 'state feedback' and 'output injection' matrices as

$$F := \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} := -R^{-1}\left[D_{1\bullet}'C_1 + B'X_\infty\right] \tag{3.6}$$

$$L := \begin{bmatrix} L_1 & L_2 \end{bmatrix} := -[B_1D_{\bullet 1}' + Y_\infty C']\tilde{R}^{-1} \tag{3.7}$$

3.1 Problem FI: Full Information

In the FI special problem G has the following form.

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] \tag{3.8}$$

It is seen that the controller is provided with *Full Information* since $y = \begin{pmatrix} x \\ w \end{pmatrix}$. In some cases, a suboptimal controller may exist which uses just the state feedback x, but this will not always be possible. While the state feedback problem is more traditional, we believe that the full information problem is more fundamental and more natural than the state feedback problem, once one gets outside the pure $\mathcal{H}_2$ setting.

The assumptions relevant to the FI problem which are inherited from the output feedback problem are

(A1) (A, B_2) is stabilizable.

(A2) D_{12} is full column rank with $\begin{bmatrix} D_{12} & D_\perp \end{bmatrix}$ unitary.

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

The results for the Full Information case are as follows:

Theorem 3.1 *Suppose G is given by (3.8) and satisfies A1-A3. Then*

(a) $\exists K$ *such that* $\|T_{zw}\|_\infty < 1 \Leftrightarrow H_\infty \in dom(Ric),\ Ric(H_\infty) \geq 0$

(b) If $\bar{\sigma}(D'_\perp D_{11}) < 1$ *then* $\exists K$ *such that* $\|T_{zw}\|_\infty \leq 1 \Leftrightarrow H_\infty \in dom(\overline{Ric}),\ X_1'X_2 = X_2'X_1 \geq 0$. *$X_1$ and X_2 are defined in (3.3).*

(c) All admissible $K(s)$ such that $\|T_{zw}\|_\infty < 1$ are given by

$$K(s) = \begin{bmatrix} -Q(s) & I \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ T_2 & I \end{bmatrix} \begin{bmatrix} F_1 & -I \\ F_2 & 0 \end{bmatrix}$$

for $Q \in \mathcal{R}_c\mathcal{H}_\infty$, $\|Q\|_\infty < 1$.

Note that the sufficiency proof for part (b) is omitted. We will prove the FI results and the FC results follow by duality.

3.2 Motivation of the proofs for Problem FI

We will first motivate the proof by considering a completion of the squares assuming that $X_\infty \geq 0$ and exists. Let us factor

$$R = \begin{bmatrix} T_1' & T_2' \\ 0 & I \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ T_2 & I \end{bmatrix} \tag{3.9}$$

$$=: T'JT \tag{3.10}$$

$$= \begin{bmatrix} -T_1'T_1 + T_2'T_2 & T_2' \\ T_2 & I \end{bmatrix} = \begin{bmatrix} D_{11}'D_{11} - I & D_{11}'D_{12} \\ D_{12}'D_{11} & I \end{bmatrix} \tag{3.11}$$

$$\Rightarrow T_2 = D_{12}'D_{11}, \quad T_1'T_1 = I - D_{11}'D_\perp D_\perp' D_{11} \tag{3.12}$$

Now $I - D_{11}'D_\perp D_\perp' D_{11} > 0$ since at $s = \infty$

$$\mathcal{F}_l(G,K)(\infty) = D_{11} + D_{12}K(\infty)\begin{bmatrix} 0 \\ I \end{bmatrix}$$

and $1 > \overline{\sigma}(\mathcal{F}_l(G,K)(\infty)) \geq \overline{\sigma}(D_\perp' D_{11})$.

Now consider the Riccati equation for X_∞,

$$\begin{bmatrix} X_\infty & -I \end{bmatrix} H_\infty \begin{bmatrix} I \\ X_\infty \end{bmatrix} = 0$$

$$\Rightarrow X_\infty A + A'X_\infty + C_1'C_1 - F'RF = 0 \tag{3.13}$$

and observe that

$$\begin{aligned}
&\left(\begin{bmatrix} w \\ u \end{bmatrix} - Fx\right)' R \left(\begin{bmatrix} w \\ u \end{bmatrix} - Fx\right) \\
&= \begin{bmatrix} w' & u' \end{bmatrix} D_{1\bullet}'D_{1\bullet} \begin{bmatrix} w \\ u \end{bmatrix} - w'w + x'(C_1'D_{1\bullet} + X_\infty B)\begin{bmatrix} w \\ u \end{bmatrix} \\
&\quad + \begin{bmatrix} w' & u' \end{bmatrix}(D_{1\bullet}'C_1 + B'X_\infty)x + x'F'RFx \\
&= z'z - x'C_1'C_1x - w'w + x'X_\infty B\begin{bmatrix} w \\ u \end{bmatrix} + \begin{bmatrix} w' & u' \end{bmatrix} B'X_\infty x + x'F'RFx \\
&= z'z - w'w + x'X_\infty Ax + x'A'X_\infty x + x'X_\infty B\begin{bmatrix} w \\ u \end{bmatrix} + \begin{bmatrix} w' & u' \end{bmatrix} B'X_\infty x \\
&= z'z - w'w + \tfrac{d}{dt}(x'X_\infty x)
\end{aligned}$$

Integrating from $t = 0$ to ∞ with $x(0) = x(\infty) = 0$ gives

$$\|z\|_2^2 - \|w\|_2^2 = \|T_2 w + u - \begin{bmatrix} T_2 & I \end{bmatrix} Fx\|_2^2 - \|T_1(w - F_1 x)\|_2^2. \tag{3.14}$$

Hence to obtain $\|z\|_2 < \|w\|_2$ we require $\|T_2 w + u - \begin{bmatrix} T_2 & I \end{bmatrix} Fx\|_2 < \|T_1(w - F_1 x)\|_2$ and we see that in some sense the "worst w" is $F_1 x$, whereas the "best u" is $-T_2 w + \begin{bmatrix} T_2 & I \end{bmatrix} Fx$. Notice that in the case $T_2 \neq 0$ ($\Rightarrow D_{11} \neq 0$) the natural full information controller uses both w and x.

3.3 Proofs for Problem FI: Necessity

(a) If there exists an admissible controller such that $\|T_{zw}\|_\infty < 1$, *then*

$$H_\infty \in dom(Ric), \quad Ric(H_\infty) \geq 0 \tag{3.15}$$

(b) If there exists an admissible controller such that $\|T_{zw}\|_\infty \leq 1$, *then*

$$H_\infty \in dom(\overline{Ric}), \quad X_1' X_2 = X_2' X_1 \geq 0. \tag{3.16}$$

We will prove a slightly stronger result, but before that, we need some preliminary results. Let us first consider

$$H_\infty = \begin{bmatrix} A & 0 \\ -C_1' C_1 & A' \end{bmatrix} - \begin{bmatrix} B \\ -C_1' D_{1\bullet} \end{bmatrix} T^{-1} J^{-1} T'^{-1} \begin{bmatrix} D_{1\bullet}' C_1, & B' \end{bmatrix}$$

where T and J are given in (3.10). Note that

$$\begin{aligned} D_{1\bullet} T^{-1} &= \begin{bmatrix} D_\perp D_\perp' D_{11} T_1^{-1}, & D_{12} \end{bmatrix} \\ B T^{-1} &= \begin{bmatrix} (B_1 - B_2 T_2) T_1^{-1}, & B_2 \end{bmatrix} =: \begin{bmatrix} \tilde{B}_1 & B_2 \end{bmatrix} \qquad (3.17) \\ I - D_{1\bullet} R^{-1} D_{1\bullet}' &= I - D_{12} D_{12}' + D_\perp D_\perp' D_{11} T_1^{-1} T_1'^{-1} D_{11}' D_\perp D_\perp' \\ &= D_\perp (I + D_\perp' D_{11} (T_1' T_1)^{-1} D_{11}' D_\perp) D_\perp' \\ &= D_\perp S^{-1} D_\perp' \qquad (3.18) \end{aligned}$$

where

$$S := I - D_\perp' D_{11} D_{11}' D_\perp > 0.$$

Hence

$$H_\infty = \begin{bmatrix} N & -B_2 B_2' + \tilde{B}_1 \tilde{B}_1' \\ -C_1' D_\perp S^{-1} D_\perp' C_1 & -N' \end{bmatrix}$$

where

$$N := A - B_2 D_{12}' C_1 + \tilde{B}_1 T_1'^{-1} D_{11}' D_\perp D_\perp' C_1$$

Next we will show that we can assume without loss of generality that the pair $(D_\perp' C_1, -N)$ is detectable. This simplifies the technical details of the proof. Thus suppose that the pair $(D_\perp' C_1, -N)$ is not detectable or equivalently that $(D_\perp' C_1, -A + B_2 D_{12}' C_1)$ is not detectable. That is, $(A - B_2 D_{12}' C_1)$ has stable modes that are not observable from $D_\perp' C_1$ (note that modes of $(A - B_2 D_{12}' C_1)$ on the imaginary axis are observable from $D_\perp' C_1$ by *A3*). If we now change state coordinates so that

$$\left[\begin{array}{c|c} A & B \\ \hline C_1 & D_{1\bullet} \end{array}\right] = \left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{11} & B_{21} \\ A_{21} & A_{22} & B_{12} & B_{22} \\ \hline C_{11} & C_{12} & D_{11} & D_{12} \end{array}\right]$$

with $A_{12} - B_{21}D'_{12}C_{12} = 0$, $D'_{\perp}C_{12} = 0$, $(D'_{\perp}C_{11}, -A_{11} + B_{21}D'_{12}C_{11})$ detectable and $(A_{22} - B_{22}D'_{12}C_{12})$ stable, then the state equations for the system with controller $K = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$ can be written as

$$\begin{aligned}
\dot{x}_1 &= A_{11}x_1 + B_{11}w + B_{21}(D'_{12}C_{12}x_2 + u) \\
z &= C_{11}x_1 + D_{11}w + D_{12}(D'_{12}C_{12}x_2 + u) \\
\dot{x}_2 &= A_{22}x_2 + A_{21}x_1 + B_{12}w + B_{22}u \\
\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}_1x_1 + \hat{B}_2x_2 + \hat{B}_3w \\
u + D'_{12}C_{12}x_2 &= \hat{C}\hat{x} + \hat{D}_1x_1 + \hat{D}_2x_2 + \hat{D}_3w + D'_{12}C_{12}x_2
\end{aligned}$$

If the controller, K, is admissible with $\|\mathcal{F}_\ell(G,K)\|_\infty < 1$ (≤ 1), then the above state equations show that the subsystem $G_1 = \left[\begin{array}{c|cc} A_{11} & B_{11} & B_{21} \\ \hline C_{11} & D_{11} & D_{12} \end{array}\right]$ also has an admissible controller, K_1 (given by the final three equations above), which satisfies $\|\mathcal{F}_\ell(G_1,K_1)\|_\infty < 1$ (≤ 1). Furthermore, suppose we can find a suitable stable invariant subspace

$$\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix}$$

for the Hamiltonian for G_1 then

$$\begin{bmatrix} X_{11} & 0 \\ 0 & I \\ X_{21} & 0 \\ 0 & 0 \end{bmatrix}$$

will be suitable for G since $(A_{22} - B_{22}D'_{12}C_{12})$ is stable. We will therefore assume that $(D'_{\perp}C_1, -A + B_2D'_{12}C_1)$ is detectable for the remainder of the necessity proof.

The proof also requires a preliminary change of variables to

$$\nu := u - F_0x$$

This change of variables will neither change internal stability nor the achievable norm since the states can be measured. The matrix F_0 is the optimal state feedback matrix for a corresponding $\mathcal{H}_2$ problem as given below. By Lemma 2.3 the Hamiltonian matrix

$$H_0 := \begin{bmatrix} A - B_2D'_{12}C_1 & -B_2B'_2 \\ -C'_1D_{\perp}D'_{\perp}C_1 & -(A - B_2D'_{12}C_1)' \end{bmatrix}$$

belongs to *dom(Ric)* since (A, B_2) is stabilizable, and $X_0 := Ric(H_0) > 0$ since $(D_\perp' C_1, -A + B_2 D_{12}' C_1)$ is detectable. Define

$$F_0 := -(D_{12}'C_1 + B_2'X_0), \quad A_{F_0} := A + B_2 F_0, \quad C_{1F_0} := C_1 + D_{12}F_0$$

$$G_c(s) := \left[\begin{array}{c|c} A_{F_0} & B_1 \\ \hline C_{1F_0} & D_{11} \end{array}\right]$$

Suppose $D_\perp$ is any matrix making $[D_{12}\ D_\perp]$ an orthogonal matrix, and define

$$\left[\begin{array}{cc} U & U_\perp \end{array}\right] = \left[\begin{array}{c|cc} A_{F_0} & B_2 & -X_0^{-1}C_1'D_\perp \\ \hline C_{1F_0} & D_{12} & D_\perp \end{array}\right] \tag{3.19}$$

Then the transfer function from w and ν to z becomes

$$z = \left[\begin{array}{c|cc} A_{F_0} & B_1 & B_2 \\ \hline C_{1F_0} & D_{11} & D_{12} \end{array}\right] \left[\begin{array}{c} w \\ \nu \end{array}\right] = G_c w + U\nu \tag{3.20}$$

The last result needed for the proof is the following lemma which is easily proven using Lemma 2.8 by obtaining a state-space realization, and then eliminating uncontrollable states using a little algebra involving the Riccati equation for X_0.

Lemma 3.2 *$[U\ U_\perp]$ is square and inner and a realization for $G_c^\sim \left[\begin{array}{cc} U & U_\perp \end{array}\right]$ is*

$$G_c^\sim \left[\begin{array}{cc} U & U_\perp \end{array}\right] = \left[\begin{array}{c|cc} A_{F_0} & B_2 & -X_0^{-1}C_1'D_\perp \\ \hline B_1'X_0 + D_{11}'C_{1F_0} & D_{11}'D_{12} & D_{11}'D_\perp \end{array}\right] \in \mathcal{RH}_2 \tag{3.21}$$

This implies that U and $U_\perp$ are each inner, and both $U_\perp^\sim G_c$ and $U^\sim G_c$ are in $\mathcal{RH}_2^\perp$.

We are now ready to state and prove the main result.

Proposition 3.3 *I. If* $\sup_{w\in B\mathcal{L}_{2+}} \min_{\nu\in\mathcal{L}_{2+}} \|z\|_2 < 1$ *then $H_\infty \in dom(Ric)$ and $Ric(H_\infty) > 0$.*

II. If $\sup_{w\in B\mathcal{L}_{2+}} \min_{\nu\in\mathcal{L}_{2+}} \|z\|_2 \le 1$ *then $H_\infty \in dom(\overline{Ric})$ and $X_1'X_2 = X_2'X_1 \ge 0$. X_1 and X_2 are defined in (3.3).*

Proof of Proposition Since $[U\ U_\perp]$ is square and inner by Lemma 3.2, $\|z\|_2 = \|[U\ U_\perp]^\sim z\|_2$, and

$$\left[\begin{array}{cc} U & U_\perp \end{array}\right]^\sim z = \left[\begin{array}{c} U^\sim G_c w + \nu \\ U_\perp^\sim G_c w \end{array}\right]$$

Since $\nu \in \mathcal{H}_2$, its optimal value is $\nu = -P_+ U^\sim G_c w$ and the hypotheses of the proposition imply that

$$\sup_{w\in B\mathcal{H}_2} \left\| \begin{bmatrix} P_-(U^\sim G_c w) \\ U_\perp^\sim G_c w \end{bmatrix} \right\|_2 < 1 \quad (\leq 1)$$

Mixed Hankel-Toeplitz operators of this type were considered in Section 2.3. We can define the adjoint operator $\Gamma^* : \mathcal{L}_{2+} \to \mathcal{W}$ ($\mathcal{W}$ from (2.14)) by

$$\Gamma^* w = \begin{bmatrix} P_-(U^\sim G_c w) \\ U_\perp^\sim G_c w \end{bmatrix} = \begin{bmatrix} P_- U^\sim \\ U_\perp^\sim \end{bmatrix} G_c w$$

of the operator $\Gamma : \mathcal{W} \to \mathcal{H}_2$ given by

$$\Gamma \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = P_+(G_c^\sim(Uq_1 + U_\perp q_2)) = P_+ G_c^\sim \begin{bmatrix} U & U_\perp \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Hence

$$\sup_{q\in B\mathcal{W}} \|\Gamma q\|_2 < 1 \qquad (\leq 1)$$

This is just the condition (2.15), so from Lemmas 2.4 and 2.7 and (3.21) we have that

$$\|G_c^\sim U_\perp\|_\infty < 1 \qquad (\leq 1)$$

and hence $H_W \in dom(Ric)$ (or $H_W \in dom(\overline{Ric})$) where (substituting for the Riccati equation for X_0 and noting that $B_1'X_0 + D_{11}'C_{1F_0} = T_1'\tilde{B}_1'X_0 + D_{11}'D_\perp D_\perp' C_1$, see (3.17),

$$\begin{aligned} (H_W)_{11} &= A_{F_0} + (-X_0^{-1}C_1'D_\perp)D_\perp' D_{11}(T_1'T_1)^{-1}(T_1'\tilde{B}_1'X_0 + D_{11}'D_\perp D_\perp' C_1) \\ &= -X_0^{-1}(A - B_2 D_{12}'C_1)'X_0 - X_0^{-1}C_1'D_\perp D_\perp' C_1 \\ &\qquad -X_0^{-1}C_1'D_\perp D_\perp' D_{11} T_1^{-1}\tilde{B}_1'X_0 \\ &\qquad -X_0^{-1}C_1'D_\perp S^{-1} D_\perp' D_{11} D_{11}' D_\perp D_\perp' C_1 \\ &= -X_0^{-1}N'X_0 - X_0^{-1}C_1'D_\perp S^{-1}D_\perp' C_1 \end{aligned}$$

$$(H_W)_{12} = X_0^{-1}C_1'D_\perp S^{-1}D_\perp' C_1 X_0^{-1}$$

$$\begin{aligned} (H_W)_{21} &= -(X_0\tilde{B}_1 T_1 + C_1'D_\perp D_\perp' D_{11})(T_1'T_1)^{-1}(T_1'\tilde{B}_1'X_0 + D_{11}'D_\perp D_\perp' C_1) \\ &= -X_0\tilde{B}_1\tilde{B}_1'X_0 - N'X_0 - X_0N + X_0B_2B_2'X_0 - C_1'D_\perp S^{-1}D_\perp' C_1 \end{aligned}$$

It is now immediate that

$$H_W = T^{-1}H_\infty T \text{ where } T = \begin{bmatrix} -I & X_0^{-1} \\ -X_0 & 0 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 0 & -X_0^{-1} \\ X_0 & -I \end{bmatrix}$$

The appropriate stable invariant subspace for H_W will be $\mathrm{Im} \begin{bmatrix} I \\ W \end{bmatrix}$ and hence that for H_∞ will be

$$\operatorname{Im} T \begin{bmatrix} I \\ W \end{bmatrix} = \operatorname{Im} \begin{bmatrix} I - X_0^{-1}W \\ X_0 \end{bmatrix}$$

Moreover Lemma 2.7 will give that $\rho(WX_0^{-1}) < 1$ (≤ 1) and hence $X_0 > W$ $(X_0 \geq W)$ giving that

$$(I - X_0^{-1}W)'X_0 = X_0 - W > 0 \quad (\geq 0)$$

or

$$X_\infty = X_0(X_0 - W)^{-1}X_0 > 0$$

in case (a). This completes the necessity proof for both parts (a) and (b).
■

3.4 Proofs for Problem FI: Sufficiency

All admissible $K(s)$ such that $\|T_{zw}\|_\infty < 1$ are given by

$$K(s) = \begin{bmatrix} -Q(s) & I \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ T_2 & I \end{bmatrix} \begin{bmatrix} F_1 & -I \\ F_2 & 0 \end{bmatrix}$$

for $Q \in \mathcal{R}_c\mathcal{H}_\infty$, $\|Q\|_\infty < 1$.

Note that this contains the *if* part of *(a)*.

Before beginning the proof, we will perform a change of variables suggested by Section 3.2. Firstly change the input variable to

$$v = u + T_2 w - \begin{bmatrix} T_2 & I \end{bmatrix} Fx$$

with the corresponding controller

$$K_{\mathrm{tmp}}(s) = K(s) + \begin{bmatrix} -\begin{bmatrix} T_2 & I \end{bmatrix} F & \quad T_2 \end{bmatrix}$$

and state equations

$$\begin{aligned} \dot{x} &= A_F x + (B_1 - B_2 T_2) w + B_2 v \\ z &= C_{1F} x + D_\perp D_\perp' D_{11} w + D_{12} v \end{aligned}$$

where

$$A_F := (A + B_2 \begin{bmatrix} T_2 & I \end{bmatrix} F); \quad C_{1F} = C_1 + D_{12} \begin{bmatrix} T_2 & I \end{bmatrix} F$$

Also define the new feedback variable

$$r := T_1(w - F_1 x)$$

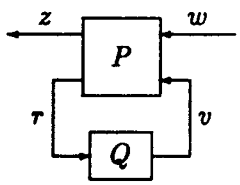
Now suppose

$$K_{\mathrm{tmp}}(s) = Q(s)T_1 \begin{bmatrix} -F_1 & I \end{bmatrix};$$

that is

$$v = Qr$$

This gives the following feedback configuration in which one would expect from (3.14) that $P^{\sim}P = I$ since $\|z\|_2^2 - \|w\|_2^2 = \|v\|_2^2 - \|r\|_2^2$ and this is now proven together with the stability of A_F.



$$P = \left[\begin{array}{c|cc} A_F & B_1 - B_2T_2 & B_2 \\ \hline C_{1F} & D_\perp D_\perp' D_{11} & D_{12} \\ -T_1F_1 & T_1 & 0 \end{array}\right] \tag{3.22}$$

Lemma 3.4 *$P \in \mathcal{R}_c\mathcal{H}_\infty$, $P^{\sim}P = I$ and $P_{21}^{-1} \in \mathcal{R}_c\mathcal{H}_\infty$.*

Proof The observability Gramian of P is X_∞ since

$$\begin{aligned} & A_F'X_\infty + X_\infty A_F + C'_{1F}C_{1F} + F'\begin{bmatrix} T_1' \\ 0 \end{bmatrix}\begin{bmatrix} T_1 & 0 \end{bmatrix}F \\ = \;& A'X_\infty + X_\infty A + C'_1C_1 + F'\begin{bmatrix} T_2' \\ I \end{bmatrix}(B_2'X_\infty + D_{12}'C_1) \\ & +(X_\infty B_2 + C_1'D_{12})\begin{bmatrix} T_2 & I \end{bmatrix}F \\ & +F'\left\{\begin{bmatrix} T_2' \\ I \end{bmatrix}\begin{bmatrix} T_2 & I \end{bmatrix} + \begin{bmatrix} T_1'T_1 & 0 \\ 0 & 0 \end{bmatrix}\right\} \\ = \;& F'\left\{-\begin{bmatrix} T_2'T_2 & T_2' \\ T_2 & I \end{bmatrix} + \begin{bmatrix} T_1'T_1 & 0 \\ 0 & 0 \end{bmatrix} + R\right\}F \\ = \;& 0 \end{aligned}$$

where we have used the identity $-B_2'X_\infty - D_{12}'C_1 = \begin{bmatrix} T_2 & I \end{bmatrix}F$. Furthermore, since $X_\infty \geq 0$ and (F_1, A_F) is detectable (note $A_F + (B_1 - B_2T_2)F_1 = A + BF$ is stable since $X_\infty = \mathrm{Ric}(H_\infty)$) we have that A_F is stable by Lemma 2.8(a). Also

$$\begin{aligned} & \begin{bmatrix} D_{11}'D_\perp D_\perp' & T_1' \\ D_{12}' & 0 \end{bmatrix}\begin{bmatrix} C_1 + D_{12}\begin{bmatrix} T_2 & I \end{bmatrix}F \\ -\begin{bmatrix} T_1, & 0 \end{bmatrix}F \end{bmatrix} + \begin{bmatrix} B_1' - T_2'B_2' \\ B_2' \end{bmatrix}X_\infty \\ = \;& \begin{bmatrix} I & -T_2' \\ 0 & I \end{bmatrix}(D_{1\bullet}'C_1 + RF + B'X_\infty) \\ = \;& 0 \end{aligned}$$

Hence by Lemma 2.8(b),

$$\begin{aligned} P^{\sim}P &= \begin{bmatrix} D_{11}'D_{\perp}D_{\perp}' & T_1' \\ D_{12}' & 0 \end{bmatrix} \begin{bmatrix} D_{\perp}D_{\perp}'D_{11} & D_{12} \\ T_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

as claimed. It is also easily shown that $P_{21}^{-1} \in \mathcal{RH}_\infty$ since its poles are $\lambda_i(A + BF)$. ■

The proof of sufficiency for Theorem 3.1(a) and the class of all controllers given in Theorem 3.1(c) can now be completed. Let K be any admissible controller such that $\|T_{zw}\|_\infty < 1$. Then $T_{vw} \in \mathcal{RH}_\infty$ and $T_{zw} = P_{11} + P_{12}T_{vw}$. Now define $Q = (I + T_{vw}P_{21}^{-1}P_{22})^{-1}T_{vw}P_{21}^{-1}$ so that $Q(I - P_{22}Q)^{-1}P_{21} = T_{vw}$ and $T_{zw} = P_{11} + P_{12}Q(I - P_{22}Q)^{-1}P_{21}$. Since P_{22} is strictly proper all the above are well-posed and Q is real-rational and proper. Hence Lemma 2.9 implies that $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < 1$. This verifies that all transfer functions T_{vw} and hence T_{uw}, can be represented in this way. ■

Remark

In the optimal case of part (b) the proof of sufficiency is more delicate and to illustrate the difficulty the following example is given. Let

$$G = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$$

then,

$$H_\infty = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T_X = -1, \quad X_1'X_2 = 0 \geq 0.$$

An optimal controller is given by

$$u = Fx - w, \Rightarrow \dot{x} = (F+1)x, \; z_1 = x, \; z_2 = u = Fx - w,$$

where $F + 1 < 0$ but F is otherwise arbitrary. Clearly for this controller $x = 0$ and hence $z_1 = 0$, $z_2 = -w$.

If the controller for the suboptimal case with $\gamma^{-2} = 1 - \epsilon^2$ is applied (see DGKF item FI.5), then,

$$\begin{aligned} X_\infty &= \frac{1 + \sqrt{1+\epsilon^2}}{\epsilon^2} \\ K(s) &= \begin{bmatrix} -X_\infty - Q(s)(1-\epsilon^2)X_\infty & Q(s) \end{bmatrix} \end{aligned}$$

An admissible optimal controller is obtained as $\epsilon \to 0$ iff $Q(s) = -1$, in which case $K(s) \to \begin{bmatrix} -(1+\sqrt{1+\epsilon^2}) & -1 \end{bmatrix}$.

3.5 Problem FC: Full Control

The FC problem has G given by,

$$G(s) = \left[\begin{array}{c|c|c} A & B_1 & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline C_1 & D_{11} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_2 & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right] \tag{3.23}$$

and is the dual of the Full Information case: the G for the FC problem has the same form as the transpose of G for the FI problem. The term *Full Control* is used because the controller has full access to both the state through output injection and to the output z. The only restriction on the controller is that it must work with the measurement y. The assumptions that the FC problem inherits from the output feedback problem are just the dual of those in the FI problem:

(A1) (C_2, A) is detectable

(A2) D_{21} is full row rank with $\begin{bmatrix} D_{21} \\ \tilde{D}_\perp \end{bmatrix}$ unitary.

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

Necessary and sufficient conditions for the FC case are given in the following corollary. The family of all controllers can be obtained from the dual of Theorem 3.1 but these will not be required in the sequel and are hence omitted.

Corollary 3.5 *Suppose G is given by (3.23) and satisfies A1, A2 and A4. Then*

(a) $\exists K$ *such that* $\|T_{zw}\|_\infty < 1 \Leftrightarrow J_\infty \in dom(Ric)$, $Ric(J_\infty) \geq 0$

(b) $\exists K$ *such that* $\|T_{zw}\|_\infty \leq 1 \Leftrightarrow J_\infty \in dom(\overline{Ric})$, $X_1'X_2 = X_2'X_1 \geq 0$. X_1 *and* X_2 *are defined in (3.4).*

4 Main Results: Output feedback

The solution to the Full Information problem of section 3 is used in this section to solve the output feedback problem. Firstly in Theorem 4.2 a so-called disturbance feedforward problem is solved. In this problem one component of the disturbance, w_2, can be estimated exactly from y using an observer, and the other component of the disturbance, w_1, does not affect the state or the output. The conditions for the existence of a controller satisfying a closed-loop $\mathcal{H}_\infty$-norm constraint is then identical to the FI case.

The solution to the general output feedback problem can then be derived from the transpose of Theorem 4.1 (Corollary 4.3) by a suitable change of variables which is based on X_∞

and the completion of the squares argument given in Section 3.2 and the characterization of all solutions given in Section 3.4.

The main result is now stated in terms of the matrices defined in section 3 involving the solutions of the X_∞ and Y_∞ Riccati equations together with the "state feedback" and "output injection" matrices F and L. It will further be convenient to additionally assume unitary changes of coordinates on w and z have been carried out to give the following partitions of D, F_1 and L_1.

$$\left[\begin{array}{c|c} & F' \\ \hline L' & D \end{array}\right] = \left[\begin{array}{c|ccc} & F'_{11} & F'_{12} & F'_2 \\ \hline L'_{11} & D_{1111} & D_{1112} & 0 \\ L'_{12} & D_{1121} & D_{1122} & I \\ L'_2 & 0 & I & 0 \end{array}\right] \tag{4.1}$$

Theorem 4.1 *Suppose G satisfies the assumptions A1–A4 of section 1.4.*

(a) There exists an admissible controller $K(s)$ such that $||\mathcal{F}_\ell(G,K)||_\infty < \gamma$ (i.e. $||T_{zw}||_\infty < \gamma$) if and only if

(i) $\gamma > max(\bar{\sigma}[D_{1111}, D_{1112},], \bar{\sigma}[D'_{1111}, D'_{1121}])$

(ii) $H_\infty \in dom(Ric)$ *with* $X_\infty = Ric(H_\infty) \geq 0$

(iii) $J_\infty \in dom(Ric)$ *with* $Y_\infty = Ric(J_\infty) \geq 0$

(iv) $\rho(X_\infty Y_\infty) < \gamma^2$.

(b) Given that the conditions of part (a) are satisfied, then all rational internally stabilizing controllers $K(s)$ satisfying $||\mathcal{F}_\ell(G,K)||_\infty < \gamma$ are given by

$$K = \mathcal{F}_\ell(K_a, \Phi) \qquad \textit{for arbitrary } \Phi \in \mathcal{R}_c\mathcal{H}_\infty \qquad \textit{such that} \quad ||\Phi||_\infty < \gamma$$

where

$$K_a = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right]$$

$$\hat{D}_{11} = -D_{1121}D'_{1111}(\gamma^2 I - D_{1111}D'_{1111})^{-1}D_{1112} - D_{1122},$$

$\hat{D}_{12} \in C^{m_2 \times m_2}$ and $\hat{D}_{21} \in C^{p_2 \times p_2}$ are any matrices (e.g. Cholesky factors) satisfying

$$\hat{D}_{12}\hat{D}'_{12} = I - D_{1121}(\gamma^2 I - D'_{1111}D_{1111})^{-1}D'_{1121},$$

$$\hat{D}'_{21}\hat{D}_{21} = I - D'_{1112}(\gamma^2 I - D_{1111}D'_{1111})^{-1}D_{1112},$$

and

$$\hat{B}_2 = Z_\infty^{-1}(B_2 + L_{12})\hat{D}_{12},$$

$$\hat{C}_2 = -\hat{D}_{21}(C_2 + F_{12}),$$

$$\hat{B}_1 = -Z_\infty^{-1} L_2 + \hat{B}_2 \hat{D}_{12}^{-1} \hat{D}_{11},$$

$$\hat{C}_1 = F_2 + \hat{D}_{11} \hat{D}_{21}^{-1} \hat{C}_2,$$

$$\hat{A} = A + BF + \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2,$$

where

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty).$$

■

(Note that if $D_{11} = 0$ then the formulae are considerably simplified.)

The proof of this main result is via some special problems that are simpler special cases of the general problem and can be derived from the FI and FC problems. A separation type argument can then give the solution to the general problem from these special problems. It can be assumed, without loss of generality, that $\gamma = 1$ since this is achieved by the scalings $\gamma^{-1}D_{11}$, $\gamma^{-1/2}B_1$, $\gamma^{-1/2}C_1$, $\gamma^{1/2}B_2$, $\gamma^{1/2}C_2$, $\gamma^{-1}X_\infty$, $\gamma^{-1}Y_\infty$ and $\gamma^{-1}K$. All the proofs will be given for the case $\gamma = 1$.

4.1 Disturbance Feedforward

In the Disturbance Feedforward problem one component of the disturbance, w_1, does not affect the state or the output. The other component of the disturbance, w_2 (and hence the state x), can be estimated exactly from y using an observer. The conditions for the existence of a controller satisfying a closed-loop $\mathcal{H}_\infty$-norm constraint is then identical to the Full Information case.

Theorem 4.2 (Disturbance Feedforward)

Theorem 4.1 is true under the additional assumptions that

$$B_1 \tilde{D}_\perp' = 0; \quad A - B_1 D_{21}' C_2 \text{ is stable.} \tag{4.2}$$

In this case,

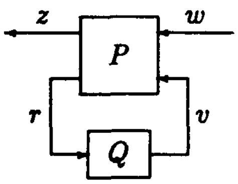
$$Y_\infty = 0,\ Z = I,\ L = -\begin{bmatrix} 0, & B_1 D_{21}' \end{bmatrix}$$

Proof

(a) The necessity of the conditions is immediate from Theorem 3.1 since the existence of an output feedback controller implies the existence of a state feedback controller. Further, the additional condition $\overline{\sigma}(D_{11}\tilde{D}'_{\perp}) < 1$ is clearly necessary by considering $s = \infty$. Theorem 3.1 also shows that all controllers satisfying $\|\mathcal{F}_\ell(G, K)\|_\infty < 1$ are given by

$$\begin{aligned} u &= Q(s)T_1(w - F_1x) + T_2(F_1x - w) + F_2x \\ r &= T_1(w - F_1x) \\ v &= Qr \end{aligned}$$

For any $Q \in R\mathcal{H}_\infty$, $\|Q\|_\infty < 1$. Also the transfer function T_{uw} is obtained from the block diagram



as $\quad T_{vw} = (I - QP_{22})^{-1}QP_{21}$ (4.3)

and hence

$$u = ((I - QP_{22})^{-1}QP_{21} - T_2)w + T_2F_1x + F_2x \tag{4.4}$$

We need to find a $Q(s)$ that can be written as an output feedback. The assumption of (4.1) will give the following realization for G,

$$G = \left[\begin{array}{c|ccc} A & 0 & B_{12} & B_2 \\ \hline C_{11} & D_{1111} & D_{1112} & 0 \\ C_{12} & D_{1121} & D_{1122} & I \\ C_2 & 0 & I & 0 \end{array}\right]$$

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Hence w_1 affects z but neither x nor y and we must firstly find a $Q(s)$ in (4.4) such that T_{uw_1} is zero. Since T_{xw_1} is zero we need

$$T_{uw_1} = (I - QP_{22})^{-1}(Q(P_{21} + P_{22}T_2) - T_2)\tilde{D}'_{\perp} = 0 \tag{4.5}$$

Using the state space realization that for $\begin{bmatrix} P_{21} & P_{22} \end{bmatrix}$ in (3.22) gives

$$[P_{21} + P_{22}T_2]\tilde{D}'_{\perp} = T_1\tilde{D}'_{\perp}$$

$$\Rightarrow QT_1\tilde{D}'_{\perp} = T_2\tilde{D}'_{\perp} \tag{4.6}$$

Again without loss of generality we can assume that

$$T_1 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{13} \end{bmatrix}$$

where

$$T_1' T_1 = I - D_{11}' D_\perp D_\perp' D_{11}$$

and hence

$$\begin{aligned} T_{11}' T_{11} &= I - D_{11\,11}' D_{11\,11} \\ T_2 \tilde{D}_\perp' &= D_{12}' D_{11} \tilde{D}_\perp' = D_{11\,21} \\ T_{11}' T_{12} &= -D_{11\,11}' D_{11\,12} \\ T_{13}' T_{13} &= I - D_{11\,12}' D_{11\,11} T_{11}^{-1} T_{11}'^{-1} D_{11\,11}' D_{11\,12} - D_{11\,12}' D_{11\,12} =: \hat{D}_{21}' \hat{D}_{21} \end{aligned}$$

Hence (4.6) implies that for $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$,

$$Q_1 = D_{11\,21} T_{11}^{-1}$$

and $QQ^\sim < I$ implies that

$$\begin{aligned} Q_2 Q_2^\sim &< I - D_{11\,21}(T_{11}' T_{11})^{-1} D_{11\,21}' \\ &= (I + D_{11\,21}(I - D_{11\,11}' D_{11\,11} - D_{11\,21}' D_{11\,21})^{-1} D_{11\,21}')^{-1} \\ &:= \hat{D}_{12} \hat{D}_{12}' \end{aligned}$$

where the indicated inverses exist by *(a)(i)*. Hence

$$Q_2 = \hat{D}_{12} Q_3 \text{ for } Q_3 \in R\mathcal{H}_\infty,\ \|Q_3\|_\infty < 1.$$

We have hence shown that all controllers can be written as feedback from w_2 and x by substituting for Q into (4.4) as

$$\begin{aligned} u &= \begin{bmatrix} D_{11\,21} T_{11}^{-1}, & \hat{D}_{12} Q_3 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{13} \end{bmatrix} \left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - F_1 x \right) \\ &\quad + \begin{bmatrix} D_{11\,21}, & D_{11\,22} \end{bmatrix} (F_1 x - w) + F_2 x \\ &= \hat{D}_{12} Q_3 \hat{D}_{21}(w_2 - D_{21} F_1 x) + D_{1121} T_{11}^{-1} [T_{12} w_2 - [T_{11} T_{12}] F_1 x] \\ &\quad + \begin{bmatrix} D_{11\,21}, & D_{11\,22} \end{bmatrix} F_1 x - D_{11\,22} w_2 + F_2 x \\ &= \hat{D}_{12} Q_3 \hat{D}_{21}(w_2 - D_{21} F_1 x) \\ &\quad + (-D_{1121}(I - D_{11\,11}' D_{11\,11})^{-1} D_{11\,11}' D_{11\,12} - D_{11\,22})(w_2 - D_{21} F_1 x) \\ &\quad + F_2 x \\ &= (\hat{D}_{11} + \hat{D}_{12} Q_3 \hat{D}_{21})(w_2 - F_{12} x) + F_2 x. \end{aligned}$$

This gives the complete family of controllers in terms of x and w_2. The disturbance, w_2, and state x can be exactly estimated from the measurement, y, by means of an observer as follows,

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + B_{12}\hat{w}_2 + B_2 u \\
\hat{w}_2 &= -C_2\hat{x} + y \\
u &= \hat{D}_{11}(\hat{w}_2 - F_{12}\hat{x}) + \hat{D}_{12}p + F_2\hat{x} \\
p &= Q_3 q \\
q &= \hat{D}_{21}\hat{w}_2 - \hat{D}_{21}F_{12}\hat{x}
\end{aligned}$$

It follows that

$$(\dot{x} - \dot{\hat{x}}) = (A - B_{12}C_2)(x - \hat{x})$$

and hence for $x(0) = \hat{x}(0) = 0$, $\hat{x}(t) = x(t)$ and $\hat{w}(t) = w(t)$ for all $t \geq 0$. Furthermore, internal stability will follow from the stability of $A - B_{12}C_2$.

Finally it is straightforward to verify that this family of controllers corresponds exactly to those of Theorem 4.1 with $Y_\infty = 0$, $Z = I$, and since

$$L = -B_1 D'_{\bullet 1}\tilde{R}^{-1} = -\begin{bmatrix} 0 & B_{12} \end{bmatrix}$$

and $0 = \mathrm{Ric}(J_\infty)$.

■

The transpose of Theorem 4.2 can now be stated to obtain another special case of Theorem 4.1.

Corollary 4.3 *(Output Estimation)*
Theorem 4.1 is true under the additional assumptions that

$$D'_\perp C_1 = 0,\ A - B_2 D'_{12} C_1 \text{ is stable.}$$

In this case

$$X_\infty = 0,\ Z = I,\ F = -\begin{bmatrix} 0 \\ D'_{12}C_1 \end{bmatrix}$$

■

4.2 Converting Output Feedback to Output Estimation

The output feedback case when the disturbance, w, cannot be estimated from the output is reduced to the case of Corollary 4.3 by a suitable change of variables. Since we showed in (3.14) that

$$\|z\|_2^2 - \|w\|_2^2 = \|v\|_2^2 - \|r\|_2^2$$

where

$$
\begin{aligned}
v &= u + T_2 w - \begin{bmatrix} T_2, & I \end{bmatrix} F x \\
r &= T_1(w - F_1 x)
\end{aligned}
$$

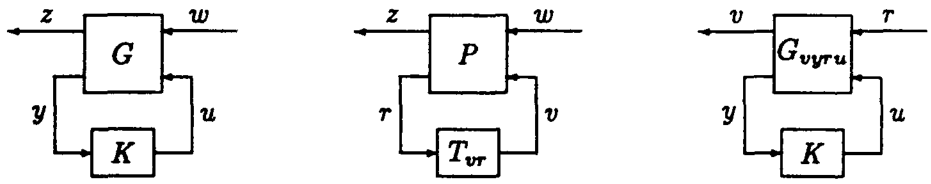
We will perform the change of variables with v replacing z and r replacing w. Hence

$$
\begin{aligned}
\dot{x} &= (A + B_1 F_1)x + B_1 T_1^{-1} r + B_2 u \\
v &= u + T_2 T_1^{-1} r - F_2 x \\
y &= C_2 x + D_{21} T_1^{-1} r + D_{21} F_1 x
\end{aligned}
$$

and the transfer matrix from $\begin{pmatrix} r \\ u \end{pmatrix}$ to $\begin{pmatrix} v \\ y \end{pmatrix}$ is

$$
G_{vyru}(s) := \left[\begin{array}{c|cc} A + B_1 F_1 & B_1 T_1^{-1} & B_2 \\ \hline -F_2 & T_2 T_1^{-1} & I \\ C_2 + D_{21} F_1 & D_{21} T_1^{-1} & 0 \end{array}\right] \tag{4.7}
$$

Similarly substituting v for u in the equation for G gives that the transfer function from $\begin{pmatrix} w \\ v \end{pmatrix}$ to $\begin{pmatrix} z \\ r \end{pmatrix}$ is P as defined in (3.22). We can show with a little algebra the equivalence of the first two of the following block diagrams, with T_{vr} given by the third one.



Lemma 4.4 *Let G satisfy A1–A4, and assume that X_∞ exists and $X_\infty \geq 0$. Then the following are equivalent:*

(a) K internally stabilizes G and $\|\mathcal{F}_\ell(G, K)\|_\infty < 1$,

(b) K internally stabilizes G_{vyru} and $\|\mathcal{F}_\ell(G_{vyru}, K)\|_\infty < 1$,

(c) K internally stabilizes G_{tmp} and $\|\mathcal{F}_\ell(G_{\text{tmp}}, K)\|_\infty < 1$,

where G_{vyru} is given by (4.7) and

$$
G_{\text{tmp}} := \left[\begin{array}{c|cc} A + B_1 F_1 & B_1 & B_2 \\ \hline -D_{12} F_2 & D_{11} & D_{12} \\ C_2 + D_{21} F_1 & D_{21} & 0 \end{array}\right].
$$

Proof

(a) $\Leftrightarrow$ **(b)** Referring to the above block diagram for P and T_{vr}, it is seen by Lemma 2.9 that $T_{zw} \in R\mathcal{H}_\infty$ with $\|T_{zw}\|_\infty < 1$ iff $T_{vr} \in R\mathcal{H}_\infty$ with $\|T_{vr}\|_\infty < 1$. (Recall that $P^\sim P = I$, $P \in R\mathcal{H}_\infty$, and $P_{21}^{-1} \in R\mathcal{H}_\infty$). In order to prove internal stability of both systems we note that this is equivalent to the realizations being stabilizable and detectable. The realization of T_{vr} is detectable since the system zeros of $(G_{vyru})_{12}$ are the eigenvalues of $A+BF$ (see Lemma 2.10). Further the realisation of T_{vr} is stabilizable from r iff the realisation of T_{zw} is stabilizable from w since they are related by state feedback. Finally if the realisation of T_{vr} is internally stable with $\|T_{vr}\|_\infty < 1$ then the above block diagram for $T_{zw} = \mathcal{F}_\ell(P, \mathcal{F}_\ell(G_{vyru}, K))$ is internally stable by a small gain argument and hence so is that for $\mathcal{F}_\ell(G, K)$.

(b) $\Leftrightarrow$ **(c)** Internal stability of both systems is equivalent since the closed-loop A-matrices are identical. Further note that

$$G_{\text{tmp}} = \begin{bmatrix} D_\perp & D_{12} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} D'_\perp D_{11} T_1^{-1} & 0 \\ \multicolumn{2}{c}{G_{vyru}} \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I \end{bmatrix}$$

and recall that

$$T_1' T_1 = I - D'_{11} D_\perp D'_\perp D_{11}.$$

Hence

$$\mathcal{F}_\ell(G_{\text{tmp}}, K) = \begin{bmatrix} D_\perp & D_{12} \end{bmatrix} \begin{bmatrix} D'_\perp D_{11} \\ \mathcal{F}_\ell(G_{vyru}, K) T_1 \end{bmatrix}$$

and

$$I - \mathcal{F}_\ell(G_{\text{tmp}}, K)^\sim \mathcal{F}_\ell(G_{\text{tmp}}, K) = T_1'(I - \mathcal{F}_\ell(G_{vyru}, K)^\sim \mathcal{F}_\ell(G_{vyru}, K)) T_1$$

hence giving the equivalence of (b) and (c).

■

The importance of the above constructions for G_{vyru} and G_{tmp} is that they satisfy the assumptions for the output estimation problem (Corollary 4.3) since $A + BF$ is stable. Hence we are now able to prove Theorem 4.1.

Proof of Theorem 4.1 (Output Feedback)

(a). The necessity of the conditions will be first proved. Let K be a proper controller satisfying $\|T_{zw}\|_\infty < 1$, then the controller $K \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$ solves the full information problem and hence (ii) holds. Similarly $\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} K$ solves the full control problem and hence (iii) holds. From Lemma 4.4 K stabilizes G_{tmp} with $\|\mathcal{F}_\ell(G_{\text{tmp}}, K)\|_\infty < 1$, which satisfies the assumptions for the output estimation problem of Corollary 4.3, since $A4$ implies that

$$\text{rank}\begin{bmatrix} A+B_1F_1-j\omega I & B_1 \\ C_2+D_{21}F_1 & D_{21} \end{bmatrix} = n+m_1$$

Hence we require

$$J_{\text{tmp}} \in dom(Ric)$$

and

$$Y_{\text{tmp}} := Ric(J_{\text{tmp}}) \geq 0$$

where

$$\begin{aligned} J_{\text{tmp}} &= \begin{bmatrix} A'+F_1'B_1' & 0 \\ -B_1B_1' & -A-B_1F_1 \end{bmatrix} \\ &\quad - \begin{bmatrix} -F_2'D_{12}' & C_2'+F_1'D_{21}' \\ -B_1D_{11}' & -B_1D_{21}' \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{11}B_1' & -D_{12}F_2 \\ D_{21}B_1' & C_2+D_{21}F_1 \end{bmatrix} \end{aligned}$$

We claim that

$$J_\alpha := \begin{bmatrix} I & -X_\infty \\ 0 & I \end{bmatrix} J_\infty \begin{bmatrix} I & X_\infty \\ 0 & I \end{bmatrix} = J_{\text{tmp}}$$

where J_∞ was defined in (3.2) as

$$J_\infty := \begin{bmatrix} A' & 0 \\ -B_1B_1' & -A \end{bmatrix} - \begin{bmatrix} C' \\ -B_1D_{\bullet 1}' \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1}B_1' & C \end{bmatrix}$$

To verify this claim let

$$\begin{aligned} M &:= \begin{bmatrix} -D_{12}F_2 \\ C_2+D_{21}F_1 \end{bmatrix} && (4.8) \\ E &:= D_{\bullet 1}B_1'X_\infty + C - M \\ \hat{N} &:= D_{1\bullet}F + C_1 \\ & && (4.9) \end{aligned}$$

Substituting for $B_1'X_\infty$ from (3.7) gives

$$\begin{aligned} B_1'X_\infty &= F_1 - D_{11}'\hat{N} && (4.10) \\ E &= D_{\bullet 1}(F_1 - D_{11}'\hat{N}) + \begin{bmatrix} C_1+D_{12}F_2 \\ -D_{21}F_1 \end{bmatrix} \\ &\quad - \begin{bmatrix} I-D_{11}D_{11}' \\ -D_{21}D_{11}' \end{bmatrix} \hat{N} \end{aligned}$$

and hence

$$- \tilde{R}^{-1}E = \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{N} \tag{4.11}$$

Now consider the claim component by component. Clearly

$$(J_\alpha)_{21} \;=\; (J_\infty)_{21} \;=\; (J_{\mathrm{tmp}})_{21}$$

Secondly

$$\begin{aligned}(J_\alpha)_{22} &= -A - B_1B_1'X_\infty + B_1D_{\bullet 1}'\tilde{R}^{-1}(M+E)\\ &= -A + B_1D_{\bullet 1}'\tilde{R}^{-1}M - B_1(B_1'X_\infty + D_{11}'\hat{N})\\ &= (J_{\mathrm{tmp}})_{22}\end{aligned}$$

by (4.10).

Finally

$$\begin{aligned}(J_\alpha)_{12} - (J_{\mathrm{tmp}})_{12} \quad &= X_\infty A + A'X_\infty + X_\infty B_1B_1'X_\infty\\ &\quad - (M'+E')\tilde{R}^{-1}(M+E) + M'\tilde{R}^{-1}M\end{aligned}$$

Substitute from (3.13):

$$\begin{aligned}X_\infty A + A'X_\infty &= -C_1'C_1 + F'RF\\ &= -(\hat{N}' - F'D_{1\bullet}')(\hat{N} - D_{1\bullet}F) + F'D_{1\bullet}'D_{1\bullet}F - F_1'F_1\\ &= -\hat{N}'\hat{N} + F'D_{1\bullet}'N + \hat{N}'D_{1\bullet}F - F_1'F_1\end{aligned}$$

Equation (4.10) gives

$$X_\infty B_1B_1'X_\infty = (F_1' - \hat{N}'D_{11})(F_1 - D_{11}'\hat{N})$$

and (4.11) and (4.8) give

$$\begin{aligned}-M'\tilde{R}^{-1}E - E'\tilde{R}^{-1}M - E'\tilde{R}^{-1}E \;&=\; -F_2'D_{12}'\hat{N} - \hat{N}'D_{12}F_2\\ &\quad - \hat{N}'(D_{11}D_{11}' - I)\hat{N}\end{aligned}$$

Adding these three expressions gives $(J_\alpha)_{12} = (J_{\mathrm{tmp}})_{12}$ and the claim that $J_\alpha = J_{\mathrm{tmp}}$ is verified.

Since

$$J_\infty \begin{bmatrix} I \\ Y_\infty \end{bmatrix} = \begin{bmatrix} I \\ Y_\infty \end{bmatrix} (A' + C'L'),$$

we have

$$J_{\mathrm{tmp}} \begin{bmatrix} I - X_\infty Y_\infty \\ Y_\infty \end{bmatrix} = \begin{bmatrix} I - X_\infty Y_\infty \\ Y_\infty \end{bmatrix} (A' + C'L')$$

and

$$Y_{\rm tmp} := {\rm Ric}(J_{\rm tmp}) = Y_\infty (I - X_\infty Y_\infty)^{-1} \geq 0.$$

It is readily verified that this implies and is implied by *(iv)*, that $\rho(X_\infty Y_\infty) < 1$. To see this, consider $Y_\infty = \begin{bmatrix} Y_{\infty 1} & 0 \\ 0 & 0 \end{bmatrix}$, $Y_{\infty 1} > 0$, and note that $Y_{\infty 1}^{-1} - X_{\infty 1} > 0$; conversely note that $X_\infty Y_\infty = (I + Y_{\rm tmp} X_\infty)^{-1} Y_{\rm tmp} X_\infty$ and hence $Y_{\rm tmp} \geq 0$ implies $\rho(X_\infty Y_\infty) < 1$.

Therefore the necessity of the condition is proven. Sufficiency also follows immediately because of the equivalence of the G and $G_{\rm tmp}$ problems.

(b) Characterization of all solutions

To characterize all controllers for G we just need to characterize all controllers for $G_{\rm tmp}$ using Corollary 4.3, with $Y_{\rm tmp} = Y_\infty Z_\infty'^{-1}$ where

$$\begin{aligned}
Z_\infty &:= (I - Y_\infty X_\infty) \\
L_{\rm tmp} &= -(B_1 D_{\bullet 1}' + Y_{\rm tmp} M')\tilde{R}^{-1} \\
&= -Z_\infty^{-1}(B_1 D_{\bullet 1}' + Y_\infty(-X_\infty B_1 D_{\bullet 1}' + M'))\tilde{R}^{-1} \\
&= -Z_\infty^{-1}(B_1 D_{\bullet 1}' + Y_\infty(C' - E'))\tilde{R}^{-1} \\
&= Z_\infty^{-1} L - Z_\infty^{-1} Y_\infty \begin{bmatrix} \hat{N}' & 0 \end{bmatrix} \\
F_{\rm tmp} &= \begin{bmatrix} 0 \\ F_2 \end{bmatrix}; \quad X_{\rm tmp} = 0; \quad Z_{\rm tmp} = I
\end{aligned}$$

We can now substitute in the formulae of Theorem 4.1 with $G_{\rm tmp}$ and the above $(\bullet)_{\rm tmp}$ values to obtain the class of controllers.

$$\begin{aligned}
\hat{B}_2 &= (B_2 + Z_\infty^{-1} L_1 D_{12} - Z_\infty^{-1} Y_\infty \hat{N}' D_{12})\hat{D}_{12} \\
&= Z_\infty^{-1}(B_2 - Y_\infty X_\infty B_2 + L_1 D_{12} - Y_\infty(F' D_{1\bullet}' + C_1')D_{12})\hat{D}_{12} \\
&= Z_\infty^{-1}(B_2 + L_1 D_{12})\hat{D}_{12}
\end{aligned}$$

by (3.7). The expressions for $\hat{C}_1$, $\hat{C}_2$, $\hat{B}_1$ and $\hat{A}$ are then obtained by a direct transcription of the above expressions and are hence omitted. This completes the proof. ■

5 Generalizations

In this section we indicate how the results of section 4 can be extended to more general cases. Firstly the optimal case is considered when a variety of new phenomena are encountered. Secondly the removal of assumptions A1–A4 is discussed. Finally some comments are included for the case when the optimal $\mathcal{H}_\infty$-norm is necessarily achieved at $s = \infty$.

5.1 The Optimal Case

In the optimal case any combination of the conditions of Theorem 4.1(a) may be violated. In order that the Hamiltonian matrices H_∞ and J_∞ can be defined we will assume that condition $(a)(i)$ in Theorem 4.1 is satisfied and will state the result proven in Glover *et al.* (1989).

Firstly if H_∞, $J_\infty \in dom(\overline{\overline{Ric}})$ then there exist $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ satisfying equation (3.3) and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ satisfying equation (3.4). In the optimal case X_1 and/or Y_1 may be singular so that $X_\infty := X_2 X_1^{-1}$ and $Y_\infty := Y_2 Y_1^{-1}$ may not exist, and if these inverses exist $Z_\infty := I - \gamma^{-2} Y_\infty X_\infty$ may be singular. In order to avoid taking these inverses we will modify the definitions of the 'state-feedback' matrix, F in (3.7), and the 'output injection' matrix, L in (3.7), as follows.

$$\begin{aligned} F^o &:= -R^{-1}\left[D_{1\bullet}' C_1 X_1 + B' X_2\right] \\ L^o &:= -[Y_1' B_1 D_{\bullet 1}' + Y_2' C'] \tilde{R}^{-1} \end{aligned}$$

Furthermore as in (4.1) we assume that D, F^o, and L^o have been transformed and partitioned as follows.

$$\left[\begin{array}{c|c} & F^{o\prime} \\ \hline L^{o\prime} & D \end{array}\right] = \left[\begin{array}{c|ccc} & F_{11}^{o\prime} & F_{12}^{o\prime} & F_2^{o\prime} \\ \hline L_{11}^{o\prime} & D_{1111} & D_{1112} & 0 \\ L_{12}^{o\prime} & D_{1121} & D_{1122} & I \\ L_2^{o\prime} & 0 & I & 0 \end{array}\right] \tag{5.1}$$

The solution to the output feedback problem in the optimal case can now be stated (Glover *et al.* (1989)).

Theorem 5.1 *Suppose G satisfies the assumptions A1–A4 of section 1.4 and*

$$\gamma > \max(\bar{\sigma}[D_{1111}, D_{1112},], \bar{\sigma}[D_{1111}', D_{1121}']).$$

(a) There exists an admissible controller $K(s)$ such that $||\mathcal{F}_\ell(G,K)||_\infty \le \gamma$ (i.e. $||T_{zw}||_\infty \le \gamma$) if and only if

(i) $H_\infty \in dom(\overline{\overline{Ric}})$ with X_1, X_2 satisfying (3.3) such that $X_1' X_2 \ge 0$.

(ii) $J_\infty \in dom(\overline{\overline{Ric}})$ with Y_1, Y_2 satisfying (3.4) such that $Y_1' Y_2 \ge 0$.

(iii) $\begin{bmatrix} X_2' X_1 & \gamma^{-1} X_2' Y_2 \\ \gamma^{-1} Y_2' X_2 & Y_2' Y_1 \end{bmatrix} \ge 0$

(b) Given that the conditions of part (a) are satisfied, then all rational internally stabilizing controllers $K(s)$ satisfying $||\mathcal{F}_\ell(G,K)||_\infty \le \gamma$ are given by

$$K = \mathcal{F}_\ell(K_a, \Phi) \quad \textit{for arbitrary } \Phi \in \mathcal{R}_c\mathcal{H}_\infty$$

$$\textit{such that} \quad ||\Phi||_\infty \le \gamma, \ \det(I - (K_a)_{22}(\infty)\Phi(\infty)) \ne 0.$$

where

$$K_a := \begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} \\ \hat{D}_{21} & 0 \end{bmatrix} + \begin{bmatrix} \hat{C}_1^o \\ \hat{C}_2^o \end{bmatrix} \left(s\hat{E} - \hat{A}\right)^{\#} \begin{bmatrix} \hat{B}_1^o & \hat{B}_2^o \end{bmatrix}$$

denotes a suitable pseudo inverse, $\hat{D}_{ij}$ are defined in Theorem 4.1. and

$$\begin{aligned}
\hat{B}_2^o &:= (Y_1'B_2 + L_{12}^o)\hat{D}_{12} \\
\hat{C}_2^o &:= -\hat{D}_{21}(C_2X_1 + F_{12}^o) \\
\hat{B}_1^o &:= -L_2^o + \hat{B}_2^o\hat{D}_{12}^{-1}\hat{D}_{11} \\
\hat{C}_1^o &:= F_2^o + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2^o \\
\hat{A}^o &:= \hat{E}T_X + \hat{B}_1^o\hat{D}_{21}^{-1}\hat{C}_2^o = T_Y'\hat{E} + \hat{B}_2^o\hat{D}_{12}^{-1}\hat{C}_1^o \\
\hat{E} &:= Y_1'X_1 - \gamma^{-2}Y_2'X_2
\end{aligned}$$

■

The descriptor form of the equations for the controllers has been used as proposed for optimal Hankel-norm approximation by Safonov *et al.* (1987). At optimality $\hat{E}$ will typically be singular and the state-space equations of Theorem 4.1 are not possible. Moreover the matrix $(s\hat{E} - \hat{A}^o)$ may be singular for all s, but the transfer function $K_a(s)$ nevertheless remains uniquely defined. The condition that $\det(I - (K_a)_{22}(\infty)\Phi(\infty)) \neq 0$ is required so that this LFT is well-posed. It is often the case that all the controllers can be characterized by $\Phi = M_1\Phi_1M_2$ for non-square constant matrices, $M_1'M_1 = I$ and $M_2M_2' = I$, with $\Phi_1 \in \mathcal{R}_c\mathcal{H}_\infty$ such that $\|\Phi_1\|_\infty \leq \gamma$.

The optimal case may also occur when H_∞ or J_∞ have eigen-values on the imaginary axis but H_∞, $J_\infty \in dom(\overline{Ric})$. In this case Theorem 5.1 can give regular state-space equations with $\hat{E}$, X_1, and Y_1 all invertible. The stable invariant subspace of H_∞ or J_∞ will only be unique when the additional constraint that $X_1'X_2$ and $Y_1'Y_2$ are Hermitian is included, and this requires some special purpose algorithms (see Section 5.2.5).

5.2 Relaxing Assumptions A1-A4

5.2.1 Relaxing A3 and A4

Suppose that,

$$G = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]$$

which violates both A3 and A4 and corresponds to the robust stabilization of an integrator. If the controller $u = -\epsilon x$, for $\epsilon > 0$ is used then

$$T_{zw} = \frac{-\epsilon s}{s+\epsilon}, \quad \text{with } \|T_{zw}\|_\infty = \epsilon$$

Hence the norm can be made arbitrarily small as $\epsilon \to 0$, but $\epsilon = 0$ is not admissible since it is not stabilizing. This may be thought of as a case where the $\mathcal{H}_\infty$-optimum is not achieved on the set of admissible controllers. Of course, for this system, $\mathcal{H}_\infty$ *optimal* control is a silly problem, although the suboptimal case is not obviously so.

If one simply drops the requirement that controllers be admissible and removes assumptions A3 and A4, then the formulae in this paper will yield $u = 0$ for both the optimal controller and the suboptimal controller with $\Phi = 0$. This illustrates that assumptions A3 and A4 are necessary for the techniques in this paper to be directly applicable. An alternative is to develop a theory which maintains the same notion of admissibility, but relaxes A3 and A4. The easiest way to do this would be to pursue the suboptimal case introducing ϵ perturbations so that A3 and A4 are satisfied.

5.2.2 Relaxing A1

If assumption A1 is violated, then it is obvious that no admissible controllers exist. Suppose A1 is relaxed to allow unstabilizable and/or undetectable modes on the $j\omega$ axis, and internal stability is also relaxed to also allow closed-loop $j\omega$ axis poles, but A2-A4 is still satisfied. It can be easily shown that under these conditions the closed-loop $\mathcal{H}_\infty$ norm cannot be made finite, and in particular, that the unstabilizable and/or undetectable modes on the $j\omega$ axis must show up as poles in the closed-loop system.

5.2.3 Violating A1 *and* either or both of A3 and A4

Sensible control problems can be posed which violate A1 *and* either or both of A3 and A4. For example, cases when A has modes at $s = 0$ which are unstabilizable through B_2 and/or undetectable through C_2 arise when an integrator is included in a weight on a disturbance input or an error term. In these cases, either A3 or A4 are also violated, or the closed-loop $\mathcal{H}_\infty$ norm cannot be made finite. In many applications such problems can be reformulated so that the integrator occurs inside the loop (essentially using the internal model principle), and is hence detectable and stabilizable.

An alternative approach to such problems which could potentially avoid the problem reformulation would be pursue the techniques in this paper, but relax internal stability to the requirement that all closed-loop modes be in the closed left half plane. Clearly, to have finite $\mathcal{H}_\infty$ norm these closed-loop modes could not appear as poles in T_{zw}. The formulae given in this paper will often yield controllers compatible with these assumptions. The user would then have to decide whether closed-loop poles on the imaginary axis were due to weights and hence acceptable or due to the problem being poorly posed as in the above example.

A third alternative is to again introduce ϵ perturbations so that A1, A3 and A4 are satisfied. Roughly speaking, this would produce sensible answers for sensible problems, but the behaviour as $\epsilon \to 0$ could be problematic.

5.2.4 Relaxing A2

In the cases that either D_{12} is not full column rank or D_{21} is not full row rank then improper controllers can give bounded $\mathcal{H}_\infty$-norm for T_{zw}, although will not be admissible as defined in section 1.4. Such singular filtering and control problems have been well-studied in $\mathcal{H}_2$ theory and many of the same techniques go over to the $\mathcal{H}_\infty$-case (e.g. Willems(1981), Willems *et* *al.*(1986) and Hautus and Silverman(1983)). In particular the structure algorithm of Silverman (1969) could be used to make the terms D_{12} and D_{21} full rank by the introduction of suitable differentiators in the controller.

5.2.5 Behaviour at $s = \infty$

It has been assumed in Theorem 5.1 that

$$\gamma > \max(\bar{\sigma}[D_{1111}, D_{1112},], \bar{\sigma}[D'_{1111}, D'_{1121}])$$

and a necessary condition for a solution is that this holds with $\geq$. If equality holds then one or both of the Hamiltonian matrices cannot be defined. This corresponds to the case

$$\inf_{K(\infty)} \bar{\sigma}(\mathcal{F}_\ell(G(\infty), K(\infty))) = 1$$

where $K(\infty)$ is just considered to be an arbitrary matrix. If

$$\inf_{K(j\omega)} \bar{\sigma}(\mathcal{F}_\ell(G(j\omega), K(j\omega))) < 1, \quad \text{for some } \omega = \omega_o$$

then a bilinear transformation from the right half plane to the right half plane that moves the point $j\omega_o$ to ∞ will enable the Hamiltonians to be defined. One of them will however have an eigen value at the point on the imaginary axis to which the point at ∞ has been transformed.

A more intricate situation arises when

$$\inf_{K(j\omega)} \bar{\sigma}(\mathcal{F}_\ell(G(j\omega), K(j\omega))) = 1, \quad \forall\, \omega.$$

Here the corresponding J-factorization problem (see Green *et al.*(1988)) or spectral factorization problem is rank deficient for all ω. The theory of spectral factorization for such cases can be derived via the solutions to a Linear Matrix Inequality (Willems(1971)), or via the stable deflating subspace of the zero pencil (see Van Dooren (1981) and Clements and Glover(1989)).

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Mass-Properties Variations in Space Station Dynamics

OUTLINE

Introduction

The space station is acted upon by internal torques and external torques. The internal torques are due to moving payloads, astronauts, control moment gyros (CMGs), etc. The external torques are due to aero, gravity gradient, earth's magnetic field, reaction jets, etc.

If there were no external torques, the total torque on the system could be kept zero by commanding the CMG torque to offset the other internal torques. The resulting CMG momentum would remain finite since the momentum of the other internal torque producing elements are finite.

To keep the system momentum bounded, the average external torques must also be zero. This could be done by using the reaction jets, however they use expendable fuel. In order to minimize the use of expendable fuel, some other source of external torque must be used to counter the aero torque. The large moments-of-inertia (MOI) of the space station give rise to attitude dependent gravity gradient torques large enough to offset the aero torque.

The gravity gradient torques depend on the MOI of the vehicle, so variations of the MOI can have a significant impact on the closed loop system stability and performance.

Several earlier studies of the space station attitude dynamics have looked at ways of analyzing the mass properties variations which appear in the moment-of-inertia matrix [WBWGLS], [BWGS], [WW], [RS], [BP]. Since the variations are quite large, it is important to remove as much conservatism as possible by taking advantage of the known structure of the moment-of-inertia matrix. The structure of a generic moment-of-inertia matrix is completely described by the fact that it is symmetric positive definite and the sum of any two of its eigenvalues is equal to or greater than the remaining eigenvalue. More specific structure is implied if the mass-properties variations arise from some specific scenario such as a single payload mass moving in some prescribed bounded region.

In this paper, three cases are examined in detail. The first case exactly represents the perturbed system dynamics for all moment-of-inertia variations due to a payload of fixed mass moving in a given rectangular region. The second case exactly represents the perturbed system dynamics for all possible moment-of-inertia matrices. The third case exactly represents the perturbed system dynamics for all diagonal moment-of-inertia matrices. Case 1 can be combined with either case 2 or case 3.

Expressing the Mass-Properties Perturbations as Linear Fractional Transformations

We have been using μ (structured singular values) to analyze the robustness of the space station controllers. In order to use the μ-synthesis or μ-analysis techniques on a perturbed linear system, the perturbations must appear as rational functions of some undetermined parameters δ. Then linear fractional transformations can be used to put the perturbed system into the standard Δ, P, K format used in the μ computations [ZD]. This procedure is related to the factorization of matrix polynomials in many δ variables and can lead to very large dimensions of the Δ matrix (especially if the factorization is not minimal).

The linearized angular dynamics of a rigid body are rational functions of the moment-of-inertia matrix J, so if the perturbed J can be written as a rational function, then the perturbed dynamics can be put into the form required for the μ computations.

The variation in the moment-of-inertia matrix J can come from several different sources. The primary variations in J come from movable payload masses and from variations in the mass properties of the space station itself.

Three types of variations in J are considered. Case 1 considers variations due to a movable payload. This type of variation is quadratic in the payload location and linear in the payload mass, so they come naturally in the form required for μ computations. A detailed description is given for how to represent this perturbed system in the Δ, P, K format. Care was taken to find the smallest possible dimension for the required Δ matrix, however the resulting Δ matrix was still 30×30. This perturbation structure was used to compute μ for various space station attitude controllers and produced reasonable results.

Case 2 considers generic variation in the J matrix. Let J be Represented in a factored form: $J = U \Sigma U^T$, where U is a special (det = 1) orthogonal matrix and Σ is a positive diagonal matrix. The physical origin of the J matrix ensures that the entries of the Σ matrix are linear in the total mass and quadratic in the geometric distribution of the mass. In order to express the entire J matrix as a rational function, we must also be able to express the U matrix as a rational function of its three independent parameters. This is made possible by the Cayley Transform. Details of representing generic variations in J as rational functions are given. The smallest size Δ matrix for this generic case is approximately 160×160 which we considered to be too large for practical use. However if the J matrix is assumed to be diagonal, then the problem simplifies enough to be of practical use.

Case 3 considers all possible diagonal perturbations to the J matrix.

NOTATION

MASS PROPERTIES

$\underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$	The vector from the space station c.g. to the payload c.g.
$\underline{\tilde{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$	
m_1	The mass of the space station (without payload)
m_2	The mass of the payload
m	The reduced mass of the space station and payload
J_1	The moment-of-inertia matrix of the space station (without payload)
J_2	The moment-of-inertia matrix of the payload
$J_{12} = -m\underline{\tilde{r}}^2$	The increase in MOI due to separation of the space-station and payload c.g.'s
J	The system moment-of-inertia matrix with respect to the system c.g.
$\begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{xy} & J_{yy} & J_{yz} \\ J_{xz} & J_{yz} & J_{zz} \end{bmatrix}$	The components of the system moment-of-inertia matrix

STATES AND INPUTS

All vectors without superscripts are in body axes. Vectors with an LV superscript are in the Local Vertical Local Horizontal reference frame which is centered in the spacecraft and rotates at orbit rate (x axis along the flight path, z axis towards the earth, y axis perpendicular to the orbit plane).

$\underline{\omega}$	angular rate vector
$C = [\underline{e}_x\ \underline{e}_y\ \underline{e}_z]$	rotation matrix from the LVLH reference frame to the body axes reference frame.
$\underline{H}$	CMG momentum vector
$\underline{T}_{aero}, \underline{T}_{CMG}$	Aero and CMG torques

LINEARIZED STATES AND INPUTS

At equilibrium, the principle body axes will not be aligned with LVLH, so define a new set of body axes which are aligned with LVLH at equilibrium. The MOI matrix, J, will not be diagonal in these new body axes. The set of three angles, $\underline{\theta}$ are the small deviations from equilibrium.

$\dot{\underline{\theta}}$	
$\underline{\theta}$	small angle deviations between the LVLH and body reference frames (not a vector)
$\underline{h}$	linearized momentum
$\underline{\tau}_{aero}, \underline{\tau}_{CMG}$	linearized aero and CMG torques

MISCELLANEOUS

I_n An $n \times n$ identity matrix

$\underline{\omega}_0^{LV} = \begin{bmatrix} 0 \\ -\omega_0 \\ 0 \end{bmatrix}$ angular rate of the LVLH reference frame (orbital rate)

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ a unit vector along the flight path in LVLH coordinates

$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ a unit vector normal to the orbit plane in LVLH coordinates

$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ a unit vector pointing towards earth in the LVLH reference frame

Nonlinear Equations of Motion

The nonlinear equations of motion for the space station and a moving payload have been examined in [LSS] and [WHS]. These references examined the results of various payload motions using non-linear simulations. In this discussion, we will assume that the speeds and accelerations of the payload are small, however the separation of the space station c.g. and the payload c.g. can be quite large so the resulting gravity gradient torques will be quite large. In later sections, we will show how to represent the linearized dynamics as a linear fractional transformation in the (x,y,z) position of the payload. This will allow us to use structured singular values to determine how robust a control system is to the MOI variations.

diagram of earth and 2-body SSF/payload

If the equations of motion are written with respect to the composite system c.g., then the translational and rotational dynamics decouple. The equations are further simplified by assuming that the two bodies have the same angular velocity.

The torque due to gravity gradient is given by:

$$\underline{T}_{GG} = 3\omega_0^2\, \underline{\tilde{e}}_z\, J\, \underline{e}_z \tag{1}$$

The torque due to payload velocity and acceleration is given by:

$$\underline{T}_{\dot{r},\ddot{r}} = m\tilde{\underline{r}}\left[2\tilde{\underline{\omega}}\,\dot{\underline{r}} - \ddot{\underline{r}}\right] \tag{2}$$

We will assume that payload motion with respect to the main body is usually quite slow, so that $\underline{T}_{\dot{r},\ddot{r}}$ can be represented as a bounded external disturbance. However the MOI matrix, J, depends on the payload position $\underline{r}$ and the resulting changes in the gravity gradient torques can be quite large.

We will assume that the aero torque can be represented as a bounded external disturbance.

The attitude dynamics are given by:

$$J\,\dot{\underline{\omega}} + \tilde{\underline{\omega}}\,J\,\underline{\omega} = 3\omega_0^2\,\tilde{\underline{e}}_z\,J\,\underline{e}_z - \underline{T} \tag{3}$$

where

$$\underline{T} = \underline{T}_{CMG} - \underline{T}_{\dot{r},\ddot{r}} - \underline{T}_{aero} \tag{4}$$

The attitude kinematics are given by:

$$\dot{C} + (\tilde{\underline{\omega}} - \tilde{\underline{\omega}}_0)\,C = 0 \qquad \text{where } C = [\underline{e}_x, \underline{e}_y, \underline{e}_z] \tag{5}$$

The CMG dynamics are given by:

$$\dot{\underline{H}} + \tilde{\underline{\omega}}\,\underline{H} = \underline{T}_{CMG} \tag{6}$$

Linearized Equations of Motion

Note that at equilibrium, the principle body axes can have arbitrary orientation with respect to the LVLH reference frame, so we will define a new set of body axes which are aligned with LVLH at equilibrium. In this new set of axes, C = I at equilibrium, but J is not diagonal.

Since C=I at equilibrium, $\underline{\omega}_0 = -\omega_0 \underline{e}_2$, and $\underline{e}_{z_0} = \underline{e}_3$.

The equilibrium equations are:

$$0 = -\omega_0^2 \tilde{\underline{e}}_2 J\underline{e}_2 + 3\omega_0^2 \tilde{\underline{e}}_3 J\underline{e}_3 - \underline{T}_0 \tag{7}$$

solving for the equilibrium torque, we get:

$$\underline{T}_0 = -\omega_0^2 \tilde{\underline{e}}_2 J\underline{e}_2 + 3\omega_0^2 \tilde{\underline{e}}_3 J\underline{e}_3 \tag{8}$$

We will now determine the linearized equations for small variations about this equilibrium.

For first order variations from equilibrium,

$$\begin{aligned} &C \approx I + \tilde{\underline{\theta}} \qquad \text{where } \tilde{\underline{\theta}} \text{ is skew symmetric} \\ &\dot{C} \approx \frac{d}{dt}\tilde{\underline{\theta}} \end{aligned} \tag{9}$$

Substituting this into the nonlinear attitude kinematics, and keeping first order terms in $\tilde{\underline{\theta}}$ and $\dot{\tilde{\underline{\theta}}}$ gives:

$$\underline{\omega} \approx \dot{\underline{\theta}} + \underline{\omega}_0^{LV} + \tilde{\underline{\omega}}_0^{LV}\, \underline{\theta} \tag{10}$$

Differentiating this and keeping first order terms in $\underline{\theta}$ and $\dot{\underline{\theta}}$ gives:

$$\dot{\underline{\omega}} \approx \ddot{\underline{\theta}} + \tilde{\underline{\omega}}_0^{LV}\, \dot{\underline{\theta}} \tag{11}$$

We now have expressions for the first order variations of the states and their derivatives in terms of the angles and rates $\underline{\theta}, \dot{\underline{\theta}}$ between the LVLH reference frame and the body reference frame. The torque inputs must also be written as first order variations from the equilibrium torque, however, the results depend on whether we use the body reference frame or the LVLH reference frame to express these torques. Note that $\underline{T}_0 = \underline{T}_0^{LV}$.

If we use the body reference frame for the torques (as in [WBWGLS]), we get

$$\underline{T} = \underline{T}_0 + \underline{\tau}_{CMG} \tag{12}$$

If we use the LVLH reference frame for the torques, we get:

$$\underline{T} = C\, \underline{T}^{LV} \approx (I + \tilde{\underline{\theta}})(\underline{T}_0^{LV} + \underline{\tau}_{CMG}^{LV}) \approx \underline{T}_0 + \underline{\tau}_{CMG}^{LV} - \tilde{\underline{T}}_0\, \underline{\theta} \tag{13}$$

Since the $-\tilde{\underline{T}}_0$ term multiplies the first order term $\underline{\theta}$, it must be included in the A matrix of the linearized system when using the LVLH torque inputs.

The equilibrium value of the torque is given by:

$$\underline{T}_0 = \omega_0^2(3\tilde{\underline{e}}_3\, J\, \underline{e}_3 - \tilde{\underline{e}}_2\, J\, \underline{e}_2) = \omega_0^2 \begin{bmatrix} -4J_{yz} \\ 3J_{xz} \\ J_{xy} \end{bmatrix} \tag{14}$$

Note that

$$\tilde{\underline{\omega}}^{LV} = C^T\, \tilde{\underline{\omega}}\, C \tag{15}$$

Plugging the first order terms for the states and the inputs into the nonlinear attitude dynamic equations and keeping only the first order terms in $\underline{\theta}$, $\dot{\underline{\theta}}$, $\ddot{\underline{\theta}}$ and $\underline{\tau}^{LV}$ gives:

$$J\,(\ddot{\underline{\theta}} + \tilde{\underline{\omega}}_0^{LV}\dot{\underline{\theta}}) \approx -\left[\dot{\underline{\theta}} + \underline{\omega}_0^{LV} + \tilde{\underline{\omega}}_0^{LV}\,\underline{\theta}\right] \times \left[J\left[\dot{\underline{\theta}} + \underline{\omega}_0^{LV} + \tilde{\underline{\omega}}_0^{LV}\,\underline{\theta}\right]\right] +$$
$$3\omega_0^3\left[\underline{e}_3 - \tilde{\underline{e}}_3\underline{\theta}\right] \times \left[J\left[\underline{e}_3 - \tilde{\underline{e}}_3\underline{\theta}\right]\right] - \left[\underline{T}_0 + \underline{\tau}\right]$$

$$= -\underline{\omega}_0^{LV} \times \left[J\,\underline{\omega}_0^{LV}\right] + 3\omega_0^3\,\underline{e}_3 \times \left[J\,\underline{e}_3\right] - \underline{T}_0 \qquad \text{constant terms} \tag{16a}$$
$$+ -\underline{\omega}_0^{LV} \times \left[J\left[\dot{\underline{\theta}} + \tilde{\underline{\omega}}_0^{LV}\,\underline{\theta}\right]\right] + 3\omega_0^3\,\underline{e}_3 \times \left[J\left[-\tilde{\underline{e}}_3\underline{\theta}\right]\right] \qquad \text{linear terms}$$
$$+ \left[J\,\underline{\omega}_0^{LV}\right] \times \left[\dot{\underline{\theta}} + \tilde{\underline{\omega}}_0^{LV}\,\underline{\theta}\right] - 3\omega_0^3\left[J\,\underline{e}_3\right] \times \left[-\tilde{\underline{e}}_3\underline{\theta}\right] - \underline{\tau} \qquad \text{linear terms}$$
$$+ \text{higher order terms}$$

Keeping only the linear terms gives:

$$J\,(\ddot{\underline{\theta}} - \omega_0\,\tilde{\underline{e}}_2\dot{\underline{\theta}}) \approx \begin{bmatrix} \tilde{\underline{e}}_2 & 3\tilde{\underline{e}}_3 & -I & 3I \end{bmatrix} \begin{bmatrix} J & 0 & 0 & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & J\tilde{\underline{e}}_2 & 0 \\ 0 & 0 & 0 & J\tilde{\underline{e}}_3 \end{bmatrix} \begin{bmatrix} I & -\tilde{\underline{e}}_2 \\ 0 & -\tilde{\underline{e}}_3 \\ I & -\tilde{\underline{e}}_2 \\ 0 & \tilde{\underline{e}}_3 \end{bmatrix} \begin{bmatrix} \omega_0\dot{\underline{\theta}} \\ \omega_0^2\underline{\theta} \end{bmatrix} - \underline{\tau} \tag{16b}$$

$$J\,\underline{\ddot{\theta}} = [L_{\dot{\theta}}(J), L_{\theta}(J)] \begin{bmatrix} \omega_0\,\underline{\dot{\theta}} \\ \omega_0^2\,\underline{\theta} \end{bmatrix} - \underline{\tau}_{CMG}$$

$$= [L_{\dot{\theta}}(J), L_{\theta}(J)] \begin{bmatrix} \omega_0\,\underline{\dot{\theta}} \\ \omega_0^2\,\underline{\theta} \end{bmatrix} - \left(\underline{\tau}_{CMG}{}^{LV} - \underline{\tilde{T}}_0\underline{\theta}\right) \qquad \textbf{(16c)}$$

$$= [L_{\dot{\theta}}{}^{LV}(J), L_{\theta}{}^{LV}(J)] \begin{bmatrix} \omega_0\,\underline{\dot{\theta}} \\ \omega_0^2\,\underline{\theta} \end{bmatrix} - \underline{\tau}_{CMG}{}^{LV}$$

where $L^{LV}(J)$ and $L(J)$ are the following linear functions of J.

$$L^{LV}(J) = [L_{\dot{\theta}}{}^{LV}(J), L_{\theta}{}^{LV}(J)] = \left[\left([J\,\underline{\tilde{e}}_2 - (J\,\tilde{\underline{e}}_2)] + \underline{\tilde{e}}_2 J\right), \left(3\underline{\tilde{e}}_3[J\,\underline{\tilde{e}}_3 - (J\,\tilde{\underline{e}}_3)] - \underline{\tilde{e}}_2[J\,\underline{\tilde{e}}_2 - (J\,\tilde{\underline{e}}_2)]\right)\right]$$

$$= \left[\begin{pmatrix} 0 & 2J_{yz} & (J_{xx} - J_{yy} + J_{zz}) \\ -2J_{yz} & 0 & 2J_{xy} \\ (-J_{xx} + J_{yy} - J_{zz}) & -2J_{xy} & 0 \end{pmatrix}, \begin{pmatrix} -4(J_{yy} - J_{zz}) & 4J_{xy} & -4J_{xz} \\ 3J_{xy} & -3(J_{xx} - J_{zz}) & -3J_{yz} \\ -J_{xz} & J_{yz} & J_{xx} - J_{yy} \end{pmatrix}\right] \qquad (17)$$

$$L(J) = [L_{\dot{\theta}}(J), L_{\theta}(J)] \qquad (18)$$

where

$$L_{\dot{\theta}}(J) = L_{\dot{\theta}}{}^{LV}(J) \qquad \text{skew symmetric.}$$

$$L_{\theta}(J) = L_{\theta}{}^{LV}(J) - \frac{\underline{\tilde{T}}_0}{\omega_0^2} = \left[L_{\theta}{}^{LV}(J)\right]^T \qquad (19)$$

In the nonlinear equations of motion, the gravity gradient torques were not a function of the rotation angle about the $\underline{e}_z$ vector and the gravity gradient torques had zero component in the z direction. In the linearized equations of motion we get one or the other of these attributes, depending on whether we represent the torques in the LVLH or body axes reference frame.

For $\underline{\tau}_{CMG}{}^{LV}$ inputs, the gravity gradient torque has zero component about the z axis which is aligned with the gravitational force. However, as the θ_z angle varies, the gravity gradient torque rotates around in the x,y plane.

For $\underline{\tau}_{CMG}$ inputs the gravity gradient torques are not a function of θ_z, but they have components in all three axis.

Linearization of the CMG dynamics is simpler in the LVLH reference frame:

$$\underline{\dot{h}}^{LV} = -\underline{\tilde{\omega}}_0\underline{h}^{LV} + \underline{\tau}_{CMG}{}^{LV} \qquad (20)$$

diagram of J and L(J) and integrators

Moments-of-Inertia for the Composite Vehicle

The system cg is given by:

$$\underline{r}_{cg} = \frac{m_1\, \underline{r}_{cg1} + m_2\, \underline{r}_{cg2}}{m_1 + m_2} \tag{21}$$

and the MOI of the system about the system cg is given by:

$$\begin{aligned} J &= -\int_{system} (\tilde{\underline{R}} - \tilde{\underline{r}}_{cg})^2 dm(\underline{R}) \\ &= -\int_{body1} (\tilde{\underline{R}} - \tilde{\underline{r}}_{cg1})^2 dm(\underline{R}) - \int_{body2} (\tilde{\underline{R}} - \tilde{\underline{r}}_{cg2})^2 dm(\underline{R}) - \frac{m_1\, m_2}{m_1 + m_2}(\tilde{\underline{r}}_{cg2} - \tilde{\underline{r}}_{cg1})^2 \end{aligned} \tag{22}$$

Let

$$J_1 = -\int_{body1} (\tilde{\underline{R}} - \tilde{\underline{r}}_{cg1})^2 dm(\underline{R}) \qquad J_2 = -\int_{body2} (\tilde{\underline{R}} - \tilde{\underline{r}}_{cg2})^2 dm(\underline{R}) \tag{23}$$

and

$$J_{12} = -\frac{m_1\, m_2}{m_1 + m_2}(\tilde{\underline{r}}_{cg2} - \tilde{\underline{r}}_{cg1})^2 \tag{24}$$

then

$$J = J_1 + J_2 + J_{12} \tag{25}$$

We can simplify the expression for J_{12} by setting

$$m = \frac{m_1\, m_2}{m_1 + m_2} \qquad \text{and} \qquad r = r_{cg2} - r_{cg1} \tag{26}$$

giving

$$J_{12} = -m\, \tilde{\underline{r}}^2 = m \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} \tag{27}$$

Perturbed System Dynamics due to Mass-Properties Variations

The only dependence on J in the linearized dynamics comes from the equation for $\ddot{\underline{\theta}}$.

We will now show how variations in J_1, J_2, and J_{12} can be combined.

Since L(J) and $L^{LV}(J)$ are linear in J,

$$L(J) = L(J_1) + L(J_2) + L(J_{12})$$

and (28)

$$L^{LV}(J) = L^{LV}(J_1) + L^{LV}(J_2) + L^{LV}(J_{12})$$

so the linearized dynamics can be written as

$$(J_1 + J_2 + J_{12})\,\ddot{\underline{\theta}} = \left(L(J_1) + L(J_2) + L(J_{12})\right) \begin{bmatrix} \omega_0\,\dot{\underline{\theta}} \\ \omega_0^2\,\underline{\theta} \end{bmatrix} - \underline{\tau}_{CMG}$$

or (29)

$$(J_1 + J_2 + J_{12})\,\ddot{\underline{\theta}} = \left(L^{LV}(J_1) + L^{LV}(J_2) + L^{LV}(J_{12})\right) \begin{bmatrix} \omega_0\,\dot{\underline{\theta}} \\ \omega_0^2\,\underline{\theta} \end{bmatrix} - \underline{\tau}_{CMG}^{LV}$$

diagram of J_1 J_2 J_{12} $L(J_1)$ $L(J_2)$ $L(J_{12})$ and integrators

We will assume that m_1, m_2, J_2, and ω_0 are all known constants. We will look at the following three cases for J_1 and J_{12}.

CASE 1: J_1 is a known constant, but J_{12} is perturbed (variable payload position).

CASE 2: J_{12} is a known constant, but J_1 has arbitrary perturbations (variable core body MOI).

CASE 3: J_{12} is a known constant, but J_1 has diagonal perturbations (variable core body MOI).

Note that case 1 can be combined with case 2 or case 3.

Case 1: Moment-of-Inertia Variations due to Payload Position

Putting the MOI and Torque Matrices Directly into Linear Fractional Form

J_{12} has only three independent parameters, (x,y,z), so we must write J_{12} and $L(J_{12})$ as linear fractional transformations with respect to

$$\Delta = \begin{bmatrix} \delta_x I_{nx} & & \\ & \delta_y I_{ny} & \\ & & \delta_z I_{nz} \end{bmatrix} \tag{30}$$

where I_{nx}, I_{ny}, and I_{nz} are identities of the appropriate dimensions.

The first step is to write J_{12} and $L(J_{12})$ as explicit functions of (x,y,z). Equation 27 already gives J_{12} as an explicit quadratic function of (x,y,z).

Using Equation 27 in equation 17, we can get $L^{LV}(J_{12})$ as an explicit quadratic function of x, y, z.

$$L^{LV}(J_{12}) = L^{LV}(-[\sqrt{m}\,\tilde{\underline{r}}]^2) = \left[m \begin{bmatrix} 0 & -2yz & 2y^2 \\ 2yz & 0 & -2xy \\ -2y^2 & 2xy & 0 \end{bmatrix} ,\quad m \begin{bmatrix} 4(y^2 - z^2) & -4xy & 4xz \\ -3xy & 3(x^2 - z^2) & 3yz \\ xz & -yz & y^2 - x^2 \end{bmatrix} \right] \tag{31}$$

Polynomial (or even rational) matrix functions in several variables can be put directly into the linear fractional format, however the dimension of the required Δ can be very large [ZD].

In order to write L^{LV} as a linear fractional transformation with respect to Δ, first write $\underline{r}$ as an average value plus differences times δ_i.

Assume that the payload is restricted to move in a rectangular region defined by the following set:

$$\underline{r} \in \left\{ \underline{r}_{ave} + \begin{bmatrix} x_{dif}\,\delta_x \\ y_{dif}\,\delta_y \\ z_{dif}\,\delta_z \end{bmatrix} \quad \text{with} \quad \begin{matrix} |\delta_x| \le 1.0 \\ |\delta_y| \le 1.0 \\ |\delta_z| \le 1.0 \end{matrix} \right\} \tag{32}$$

The 3×6 L^{LV} matrix polynomial in equation 31 can be written using 3×6 coefficient matrices Q_{ij}, Q_i, Q_0 which are functions of $\underline{r}_{ave}$ and x_{dif}, y_{dif}, and z_{dif}.

$$L^{LV} = \sum_{i=1}^{3}\left[\left(\sum_{j \ge i} Q_{ij}\delta_j\right) + Q_i\right]\delta_i + Q_0 \tag{33}$$

This matrix polynomial can be written as a linear fractional transformation as follows:

$$L^{LV} = \mathrm{LFT}_u\left(\begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix}, \Delta_Q\right) = D_Q + C_Q\Delta_Q(I - A_Q\Delta_Q)^{-1}B_Q \tag{34}$$

where A_Q is chosen to be nilpotent of order 1, so that $(I - A_Q\Delta_Q)^{-1} = (I + A_Q\Delta_Q)$

$$A_Q = \begin{bmatrix} 0_{3\times3} & Q_{11} & 0_{3\times3} & Q_{12} & 0_{3\times3} & Q_{13} \\ 0_{6\times3} & 0_{6\times6} & 0_{6\times3} & 0_{6\times6} & 0_{6\times3} & 0_{6\times6} \\ 0_{3\times3} & 0_{3\times6} & 0_{3\times3} & Q_{22} & 0_{3\times3} & Q_{23} \\ 0_{6\times3} & 0_{6\times6} & 0_{6\times3} & 0_{6\times6} & 0_{6\times3} & 0_{6\times6} \\ 0_{3\times3} & 0_{3\times6} & 0_{3\times3} & 0_{3\times6} & 0_{3\times3} & Q_{33} \\ 0_{6\times3} & 0_{6\times6} & 0_{6\times3} & 0_{6\times6} & 0_{6\times3} & 0_{6\times6} \end{bmatrix} \tag{35}$$

$$B_Q = \begin{bmatrix} 0_{3\times6} \\ I_{6\times6} \\ 0_{3\times6} \\ I_{6\times6} \\ 0_{3\times6} \\ I_{6\times6} \end{bmatrix} \qquad C_Q = \begin{bmatrix} I_{3\times3} & Q_1 & I_{3\times3} & Q_2 & I_{3\times3} & Q_3 \end{bmatrix} \qquad D_Q = Q_0 \tag{36}$$

$$\Delta = \mathrm{diag}(\delta_x I_9,\ \delta_y I_9,\ \delta_z I_9) \tag{37}$$

This representation of L^{LV} requires 27 δs. In addition, it takes 12 δs to represent J (or J^{-1}) as an LFT (see the section on expressing $\underline{\dot{r}}$ as an LFT). The total number of δs required is 39.

In order to reduce the dimension of the Δ, it is sometimes useful to first factor the matrix polynomials. Polynomial functions in several variables cannot always be factored into linear terms, however in our case it is possible.

Factoring the MOI and Torque Matrices

J_{12} is already in factored form (see equation 27). The 3×6 matrix $L(-[\sqrt{m}\tilde{r}]^2)$ can also be factored into left and right factors, each of which is linear in $\sqrt{m}\underline{r}$. The factors are not unique. The freedom in the factorization is parameterized by an arbitrary vector $\underline{\lambda}$ whose values are then chosen to reduce the number of δs needed in the LFT of the factor (see appendix for details).

$$L(-[\sqrt{m}\tilde{\underline{r}}]^2) = F_L(\sqrt{m}\underline{r})\, F_R(\sqrt{m}\underline{r})$$

$$L^{LV}(-[\sqrt{m}\tilde{\underline{r}}]^2) = F_L^{LV}(\sqrt{m}\underline{r})\, F_R^{LV}(\sqrt{m}\underline{r})$$

where (38)

$$F_L(\sqrt{m}\underline{r}) = \sqrt{m}\,\tilde{\underline{r}}$$

$$F_R(\sqrt{m}\underline{r}) = \sqrt{m}\begin{bmatrix} 2y & 0 & 0 & 0 & -3z & -y \\ 0 & 2y & 0 & 4z & 0 & -x \\ 0 & 0 & 2y & 4y & -3x & 0 \end{bmatrix} + \sqrt{m}\left[\underline{r}\,\underline{\lambda}_{\dot{\theta}}^T,\ \underline{r}\,\underline{\lambda}_{\theta}^T\right] \qquad \text{choose } \underline{\lambda}_{\dot{\theta}} = \underline{0} = \underline{\lambda}_{\theta}$$

$$F_L^{LV}(\sqrt{m}\underline{r}) = \sqrt{m}\begin{bmatrix} 2y & 0 & 0 & 0 & -4z & -4y \\ 0 & 2y & 0 & 3z & 0 & 3x \\ 0 & 0 & 2y & y & x & 0 \end{bmatrix} + \sqrt{m}\left[\underline{\lambda}_{\dot{\theta}}^{LV}\,\underline{r}^T,\ \underline{\lambda}_{\theta}^{LV}\,\underline{r}^T\right] \qquad \text{choose } \lambda_{\dot{\theta}} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \qquad \lambda_\theta = \underline{0}$$

$$F_R^{LV}(\sqrt{m}\underline{r}) = \begin{bmatrix} \sqrt{m}\,\tilde{\underline{r}} & 0_{3\times3} \\ 0_{3\times3} & \sqrt{m}\,\tilde{\underline{r}} \end{bmatrix} \tag{40}$$

Equation 16 now becomes:

$$\left[J_1 + J_2 - (\sqrt{m}\tilde{\underline{r}})^2\right]\ddot{\underline{\theta}} = \left[L(J_1) + L(J_2) + F_L(\sqrt{m}\underline{r})\, F_R(\sqrt{m}\underline{r})\right]\begin{bmatrix} \omega_0\,\dot{\underline{\theta}} \\ \omega_0^2\,\underline{\theta} \end{bmatrix} - \underline{\tau}_{CMG}$$

$$= \left[L^{LV}(J_1) + L^{LV}(J_2) + F_L^{LV}(\sqrt{m}\underline{r})\, F_R^{LV}(\sqrt{m}\underline{r})\right]\begin{bmatrix} \omega_0\,\dot{\underline{\theta}} \\ \omega_0^2\,\underline{\theta} \end{bmatrix} - \underline{\tau}_{CMG}^{LV} \tag{41}$$

The only dependence of the dynamics on m and $\underline{r}$ is through $\sqrt{m}\,\tilde{\underline{r}}$ and $F(\sqrt{m}\,\underline{r})$, both of which are linear in $\sqrt{m}\underline{r}$.

big diagram with J_1 J_2 $\sqrt{m}\underline{\tilde{r}}$
$L(J_1)$ $L(J_2)$ Fsys integrators

for body and LVLH torque inputs

with DELTA sizes labeled

Putting the Factors into Linear Fractional Form

Expressing $\sqrt{m}\underline{\tilde{r}}$ as a Linear Fractional Transformation

Assume that the payload is restricted to move in a rectangular region defined by the following set:

$$\underline{r} \in \left\{ \underline{r}_{ave} + \begin{bmatrix} x_{dif}\,\delta_x \\ y_{dif}\,\delta_y \\ z_{dif}\,\delta_z \end{bmatrix} \quad \text{with} \quad \begin{matrix} |\delta_x| \le 1.0 \\ |\delta_y| \le 1.0 \\ |\delta_z| \le 1.0 \end{matrix} \right\} \tag{42}$$

then

$$\underline{\tilde{r}} = \underline{\tilde{r}}_{ave} + x_{dif}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \delta_x & 0 \\ 0 & \delta_x \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + y_{dif}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \delta_y & 0 \\ 0 & \delta_y \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} + z_{dif}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \delta_z & 0 \\ 0 & \delta_z \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{43}$$

So

$$\sqrt{m}\underline{\tilde{r}} = C_{\tilde{r}}\,\Delta_{\tilde{r}}\,B_{\tilde{r}} + D_{\tilde{r}} \tag{44}$$

where

$$B_{\tilde{r}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad C_{\tilde{r}} = \sqrt{m}\begin{bmatrix} 0 & 0 & y_{dif} & 0 & z_{dif} & 0 \\ x_{dif} & 0 & 0 & 0 & 0 & z_{dif} \\ 0 & x_{dif} & 0 & y_{dif} & 0 & 0 \end{bmatrix} \qquad D_{\tilde{r}} = \sqrt{m}\,\underline{\tilde{r}}_{ave} \tag{45}$$

and

$$\Delta_{\tilde{r}} = \text{diag}[\delta_x I_2,\ \delta_y I_2,\ \delta_z I_2] \tag{46}$$

Let $P_{\tilde{r}}$ be the following "system" matrix

$$P_{\tilde{r}} = \begin{bmatrix} 0 & B_{\tilde{r}} \\ C_{\tilde{r}} & D_{\tilde{r}} \end{bmatrix} \tag{47}$$

Then equation 44 becomes

$$\sqrt{m}\,\underline{\tilde{r}} = \text{LFT}_u\left[P_{\tilde{r}},\ \Delta_{\tilde{r}}\right] \tag{48}$$

where LFT_u is a Linear Fractional Transformation

Since $J_{12} = -m(\tilde{r})^2$, the above LFT for $\underline{\tilde{r}}$ is used twice in the expression for for J_{12} on the left hand side of equation 29.

Expressing F_R as a Linear Fractional Transformation

$$
\begin{aligned}
&F_R(\sqrt{m}\,\underline{r}) = F_R(\sqrt{m}\,\underline{r}_{ave}) + \\
&\sqrt{m}\,x_{dif}\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}\delta_x&0\\0&\delta_x\end{bmatrix}\begin{bmatrix}0&0&0&0&0&-1\\0&0&0&0&-3&0\end{bmatrix} + \\
&\sqrt{m}\,y_{dif}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\delta_y&0&0\\0&\delta_y&0\\0&0&\delta_y\end{bmatrix}\begin{bmatrix}2&0&0&0&0&-1\\0&2&0&0&0&0\\0&0&2&4&0&0\end{bmatrix} + \\
&\sqrt{m}\,z_{dif}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}\delta_z&0\\0&\delta_z\end{bmatrix}\begin{bmatrix}0&0&0&0&-3&0\\0&0&0&4&0&0\end{bmatrix}
\end{aligned}
\tag{49}
$$

so

$$F_R(\sqrt{m}\,\underline{r}) = C_{F_R}\,\Delta_{F_R}\,B_{F_R} + D_{F_R} \tag{50}$$

where

$$
B_{F_R} = \begin{bmatrix}0&0&0&0&0&-1\\0&0&0&0&-3&0\\2&0&0&0&0&-1\\0&2&0&0&0&0\\0&0&2&4&0&0\\0&0&0&0&-3&0\\0&0&0&4&0&0\end{bmatrix}\quad
C_{F_R} = \sqrt{m}\begin{bmatrix}0&0&y_{dif}&0&0&z_{dif}&0\\x_{dif}&0&0&y_{dif}&0&0&z_{dif}\\0&x_{dif}&0&0&y_{dif}&0&0\end{bmatrix}\quad
D_{F_R} = F_R(\sqrt{m}\underline{r}_{ave}) \tag{51}
$$

and

$$\Delta_{F_R} = \mathrm{diag}[\delta_x I_2,\ \delta_y I_3,\ \delta_z I_2] \tag{52}$$

Let P_{F_R} be the following "system" matrix

$$P_{F_R} = \begin{bmatrix}0 & B_{F_R}\\ C_{F_R} & D_{F_R}\end{bmatrix} \tag{53}$$

Then equation 50 becomes

$$F_R(\sqrt{m}\,\underline{r}) = \mathrm{LFT}_u\left[P_{F_R},\ \Delta_{F_R}\right] \tag{54}$$

where LFT_u is a Linear Fractional Transformation

Expressing F_L^{LV} as a Linear Fr·ctional Transformation

$$
\begin{aligned}
F_L^{LV}(\sqrt{m}\,\underline{r}) &= F_L^{LV}(\sqrt{m}\,\underline{r}_{ave}) \\
&+ \sqrt{m}\,x_{dif}\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}\delta_x&0\\0&\delta_x\end{bmatrix}\begin{bmatrix}-2&0&0&0&0&3\\0&0&0&0&1&0\end{bmatrix} \\
&+ \sqrt{m}\,y_{dif}\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}\delta_y&0\\0&\delta_y\end{bmatrix}\begin{bmatrix}2&0&0&0&0&-4\\0&0&2&1&0&0\end{bmatrix} \\
&+ \sqrt{m}\,z_{dif}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}\delta_z&0\\0&\delta_z\end{bmatrix}\begin{bmatrix}0&0&0&0&-4&0\\0&0&-2&3&0&0\end{bmatrix}
\end{aligned}
\tag{55}
$$

so

$$
F_L^{LV}(\sqrt{m}\,\underline{r}) = C_{F_L^{LV}}\,\Delta_{F_L^{LV}}\,B_{F_L^{LV}} + D_{F_L^{LV}}
\tag{56}
$$

where

$$
B_{F_L^{LV}} = \begin{bmatrix}-2&0&0&0&0&3\\0&0&0&0&1&0\\2&0&0&0&0&-4\\0&0&2&1&0&0\\0&0&0&0&-4&0\\0&0&-2&3&0&0\end{bmatrix}
\qquad
C_{F_L^{LV}} = \sqrt{m}\begin{bmatrix}0&0&y_{dif}&0&z_{dif}&0\\x_{dif}&0&0&0&0&z_{dif}\\0&x_{dif}&0&y_{dif}&0&0\end{bmatrix}
\tag{57}
$$

$$
D_{F_L^{LV}} = F_L^{LV}(\sqrt{m}\,\underline{r}_{ave}) \qquad \Delta_{F_L^{LV}} = \mathrm{diag}[\delta_x I_2,\ \delta_y I_2,\ \delta_z I_2]
\tag{58}
$$

Let $P_{F_L^{LV}}$ be the following "system" matrix

$$
P_{F_L^{LV}} = \begin{bmatrix}0 & B_{F_L^{LV}}\\ C_{F_L^{LV}} & D_{F_L^{LV}}\end{bmatrix}
\tag{59}
$$

Then equation 56 becomes

$$
F_L^{LV}(\sqrt{m}\,\underline{r}) = \mathrm{LFT}_u\left[P_{F_L^{LV}},\ \Delta_{F_L^{LV}}\right]
\tag{60}
$$

where LFT_u is a Linear Fractional Transformation

Case 2: Generic Mass-Properties Variations for a Single Body

A generic moment-of-inertia matrix can be written as

$$J = U \begin{bmatrix} myy + mzz & 0 & 0 \\ 0 & mxx + mzz & 0 \\ 0 & 0 & mxx + myy \end{bmatrix} U^T \tag{61}$$

where U is an orthogonal matrix (three free parameters in U) Note that this parameterization of J automatically satisfies the constraint that the sum of any two of its eigenvalues is greater or equal to the third eigenvalue.

The Cayley Transform provides a way of parameterizing U as a rational function of three parameters. Since rational functions can be written as linear fractional transformations, they can be handled by the structured singular value algorithms.

Let $\underline{\zeta} \in \mathbf{R}^3$ and let

$$U = (I_3 + \underline{\zeta})(I_3 - \underline{\zeta})^{-1} \tag{62}$$

Then U is an orthogonal matrix, $(U^T = U^{-1})$.

Proof:

$$\begin{aligned} &(I_3 + \underline{\zeta}) \left[U^T - U^{-1}\right] (I_3 + \underline{\zeta}) \\ &= (I_3 + \underline{\zeta}) \left[(I_3 + \underline{\zeta})^{-1} (I_3 - \underline{\zeta}) - (I_3 - \underline{\zeta})(I_3 + \underline{\zeta})^{-1}\right] (I_3 + \underline{\zeta}) \\ &= (I_3 - \underline{\zeta}) - (I_3 - \underline{\zeta}) = 0 \end{aligned} \tag{63}$$

and

$$\det(I_3 + \underline{\zeta}) = 1 + |\underline{\zeta}| \neq 0 \tag{64}$$

so

$$U^{-1} - U^T = 0 \tag{65}$$

The LFTs for $(I_3 + \underline{\zeta})$ and $(I_3 - \underline{\zeta})^{-1}$ each take 6 δs, so U and U^T each take 12 deltas. Σ also takes 6 δs, so J takes a total of 30 δs. Since the expression for the dynamics in equation 16b has J in it 5 different places, it could take as many as as 150 δs for the whole system when generic moment-of-inertia matrices are used.

Case 3: Diagonal Mass-Properties Perturbations for a Single Body

In [BP], MOI variations which leave J diagonal are put into LFT form. Some of those results are reproduced here.

If $U = I$, so J is always diagonal, then the moment-of-inertia matrix can be written as

$$J = \begin{bmatrix} myy + mzz & 0 & 0 \\ 0 & mxx + mzz & 0 \\ 0 & 0 & mxx + myy \end{bmatrix} \tag{66}$$

Note that

$$mxx = \frac{J_{yy} + J_{zz} - J_{xx}}{2} \qquad myy = \frac{J_{xx} + J_{zz} - J_{yy}}{2} \qquad mzz = \frac{J_{xx} + J_{yy} - J_{zz}}{2} \tag{67}$$

For this J,

$$L^{LV}(J) = L(J) = \left[\begin{bmatrix} 0 & 0 & 2myy \\ 0 & 0 & 0 \\ -2myy & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4(myy - mzz) & 0 & 0 \\ 0 & 3(mxx - mzz) & 0 \\ 0 & 0 & myy - mxx \end{bmatrix} \right] \tag{68}$$

Since J and L(J) are both linear in the three independent parameters mxx, myy, and mzz, they can be put directly into linear fractional form.

Assume that each diag element of J lies between some maximum and some minimum.
Let

$$\begin{aligned} J_{ave} &= \frac{J_{max} + J_{min}}{2} \\ J_{dif} &= \frac{J_{max} - J_{min}}{2} \end{aligned} \tag{69}$$

Let

$$\begin{aligned} mxx_{dif} &= \frac{J_{dif_{yy}} + J_{dif_{zz}} - J_{dif_{xx}}}{2} \\ myy_{dif} &= \frac{J_{dif_{xx}} + J_{dif_{zz}} - J_{dif_{yy}}}{2} \\ mzz_{dif} &= \frac{J_{dif_{xx}} + J_{dif_{yy}} - J_{dif_{zz}}}{2} \end{aligned} \tag{70}$$

Then

$$
\begin{aligned}
J &= J_{ave} + \\
&\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_{mxx} & 0 \\ 0 & \delta_{mxx} \end{bmatrix} \begin{bmatrix} 0 & mxx_{dif} & 0 \\ 0 & 0 & mxx_{dif} \end{bmatrix} + \\
&\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_{myy} & 0 \\ 0 & \delta_{myy} \end{bmatrix} \begin{bmatrix} myy_{dif} & 0 & 0 \\ 0 & 0 & myy_{dif} \end{bmatrix} + \\
&\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{mzz} & 0 \\ 0 & \delta_{mzz} \end{bmatrix} \begin{bmatrix} mzz_{dif} & 0 & 0 \\ 0 & mzz_{dif} & 0 \end{bmatrix} \\
&= LFT_u\left(\begin{bmatrix} 0 & B_J \\ C_J & D_J \end{bmatrix}, \Delta_J\right)
\end{aligned}
\tag{71}
$$

where

$$
B_J = \begin{bmatrix} 0 & mxx_{dif} & 0 \\ 0 & 0 & mxx_{dif} \\ myy_{dif} & 0 & 0 \\ 0 & 0 & myy_{dif} \\ mzz_{dif} & 0 & 0 \\ 0 & mzz_{dif} & 0 \end{bmatrix} \qquad C_J = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad D_J = J_{ave} \tag{72}
$$

$$
\Delta_J = \mathrm{diag}(\delta_{mxx} I_2, \delta_{myy} I_2, \delta_{mzz} I_2) \tag{73}
$$

$$
\begin{aligned}
L(J) &= \begin{bmatrix} 0 & 0 & 2myy & 4(myy - mzz) & 0 & 0 \\ 0 & 0 & 0 & 0 & 3(mxx - mzz) & 0 \\ -2myy & 0 & 0 & 0 & 0 & myy - mxx \end{bmatrix} \\
&= L(J_{ave}) + \\
&\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_{mxx} & 0 \\ 0 & \delta_{mxx} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 3mxx_{dif} & 0 \\ 0 & 0 & 0 & 0 & 0 & -mxx_{dif} \end{bmatrix} + \\
&\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_{myy} & 0 \\ 0 & \delta_{myy} \end{bmatrix} \begin{bmatrix} 0 & 0 & 2myy_{dif} & 4myy_{dif} & 0 & 0 \\ -2myy_{dif} & 0 & 0 & 0 & 0 & myy_{dif} \end{bmatrix} + \\
&\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{mzz} & 0 \\ 0 & \delta_{mzz} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -4mzz_{dif} & 0 & 0 \\ 0 & 0 & 0 & 0 & -3mzz_{dif} & 0 \end{bmatrix} \\
&= LFT_u\left(\begin{bmatrix} 0 & B_{L(J)} \\ C_{L(J)} & D_{L(J)} \end{bmatrix}, \Delta_{L(J)}\right)
\end{aligned} \tag{74}
$$

where

$$
B_{L(J)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 3mxx_{dif} & 0 \\ 0 & 0 & 0 & 0 & 0 & -mxx_{dif} \\ 0 & 0 & 2myy_{dif} & 4myy_{dif} & 0 & 0 \\ -2myy_{dif} & 0 & 0 & 0 & 0 & myy_{dif} \\ 0 & 0 & 0 & -4mzz_{dif} & 0 & 0 \\ 0 & 0 & 0 & 0 & -3mzz_{dif} & 0 \end{bmatrix} \qquad C_{L(J)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{75}
$$

$$
D_{L(J)} = L(J_{ave}) \qquad \Delta_{L(J)} = \mathrm{diag}(\delta_{mxx} I_2, \delta_{myy} I_2, \delta_{mzz} I_2) \tag{76}
$$

This requires 6 δs for $L(J)$ and 6 δs for J.

Scaling the Inputs, Outputs, States, and Time

Using English units, the space station has moments of inertia on the order of $\|J\| = 10^8$ and the orbital rate is $\omega_0 = 10^{-3}$. These numbers show up raised to various powers in the state space equations and give them large condition number. When solving the Ricatti equations for the H_∞ or H_2 controllers, numerical problems arise due to the poor condition number of the state space equations. One way to dramatically improve the condition number is to make the system dimensionless. We made the equations of motions dimensionless, by substituting:

$$\omega_0 \longrightarrow 1 \tag{77}$$

and

$$J \longrightarrow \frac{J}{J_{zz}} \tag{78}$$

In order to make the perturbation equations dimensionless too, we must also substitute:

$$(\sqrt{m}\ \underline{r}) \longrightarrow \frac{(\sqrt{m}\ \underline{r})}{\sqrt{J_{zz}}} \tag{79}$$

This is equivalent to scaling time, scaling the state, and scaling the inputs.

VARIABLE	SCALE FACTOR
time	$1/\omega_0$
momentum	$J_{zz}\omega_0$
angular rate	ω_0
angles	1
torques	$J_{zz}\omega_0^2$

Appendix on Factorization of 3 × 3 Quadratic Matrix Polynomials

Generic 3×3 homogeneous quadratic matrix polynomials, $Q(\underline{r})$, in 3 variables r_1, r_2, r_3 may not be factorable, however we are concerned with a special class of such matrices which satisfy $Q(\underline{r})\,\underline{r} = \underline{0}$. This class of matrices does factor into left and right linear factor matrices: $Q(\underline{r}) = F_L(\underline{r})\,F_R(\underline{r})$, and an explicit formula is given for each factor.

THEOREM:

Given:

$\underline{r} \in \mathbf{R}^3$ (3 variables)

$\tilde{\underline{r}} \in \mathbf{R}^{3\times3}$ skew matrix formed from elements of $\underline{r}$

$q = \left[\underline{q}_{11_2}, \underline{q}_{11_3}, \underline{q}_{22_1}, \underline{q}_{22_3}, \underline{q}_{33_1}, \underline{q}_{33_2}, \underline{q}_{23_1}, \underline{q}_{13_2}\right] \in \mathbf{R}^{3\times8}$ (24 coefficients)

$\underline{\lambda} \in \mathbf{R}^3$ (3 coefficients)

$F(\underline{r}) \in \mathbf{R}^{3\times3}$ Matrix whose elements are homogeneous linear polynomials in the elements of $\underline{r}$

$Q(\underline{r}) \in \mathbf{R}^{3\times3}$ Matrix whose elements are homogeneous quadratic polynomials in the elements of $\underline{r}$

The following three statements are equivalent:

1) $Q(\underline{r})\,\underline{r} = \underline{0}$

2) $Q(\underline{r}) = F(\underline{r})\,\tilde{\underline{r}}$

3) $Q(\underline{r})$ has 24 free coefficients q, and is of the form:

$$Q(\underline{r}) = \left[\underline{0}, \underline{q}_{11_2}, \underline{q}_{11_3}\right] r_1 r_1 + \left[\underline{q}_{22_1}, \underline{0}, \underline{q}_{22_3}\right] r_2 r_2 + \left[\underline{q}_{33_1}, \underline{q}_{33_2}, \underline{0}\right] r_3 r_3 +$$

$$+ \left[\underline{q}_{23_1}, -\underline{q}_{22_3}, -\underline{q}_{33_2}\right] r_2 r_3 + \left[-\underline{q}_{11_3}, \underline{q}_{13_2}, -\underline{q}_{33_1}\right] r_1 r_3 + \left[-\underline{q}_{11_2}, -\underline{q}_{22_1}, -\underline{q}_{23_1} - \underline{q}_{13_2}\right] r_1 r_2$$

PROOF

2) --> 1)

$$Q(\underline{r})\,\underline{r} = F(\underline{r})\,\tilde{\underline{r}}\,\underline{r} = \underline{0}$$

3) --> 2)

Given any matrix of the form of statement 3), set it equal to a general 3×3 homogeneous linear matrix times $\tilde{\underline{r}}$ and then equate coefficients. A general homogeneous linear matrix polynomial in the entries of $\underline{r}$ is of the form:

$$F(\underline{r}) = F_1\, r_1 + F_2\, r_2 + F_3\, r_3$$

where $F_i \in \mathbf{R}^{3\times3}$ are coefficient matrices. Setting

$$\left[F_1\, r_1 + F_2\, r_2 + F_3\, r_3\right] \underline{r} = Q(\underline{r})$$

and equating coefficients gives:

$$F_1 = \left[\underline{\lambda} - \underline{g}_{13_2},\ -\underline{g}_{11_3},\ \underline{g}_{11_2}\right] \qquad F_2 = \left[\underline{g}_{22_3},\ \underline{\lambda} + \underline{g}_{23_1},\ -\underline{g}_{22_1}\right] \qquad F_3 = \left[-\underline{g}_{33_2},\ \underline{g}_{33_1},\ \underline{\lambda} + \underline{g}_{13_2}\right]$$

where $\underline{\lambda}$ is an arbitrary vector. Note that $F(\underline{r})$ has 27 free parameters, $q, \underline{\lambda}$ so all possible homogeneous linear matrices can be generated as factors of the quadratic matrices given in statement 3).

1) --> 3)

A general 3×3 matrix homogeneous quadratic polynomial in three variables has 54 free coefficients and is of the form:

$$Q(\underline{r}) = Q_{11}\, r_1r_1 + Q_{22}\, r_2r_2 + Q_{33}\, r_3r_3 + Q_{23}\, r_2r_3 + Q_{13}\, r_1r_3 + Q_{12}\, r_1r_2$$

where the Q_{ij} are 3×3 coefficient matrices.

Let the three columns of each Q_{ij} be denoted by:

$$Q_{ij} = \left[\underline{g}_{ij_1},\ \underline{g}_{ij_2},\ \underline{g}_{ij_3}\right]$$

then

$$\begin{aligned}
Q(\underline{r})\, \underline{r} = {} & \underline{g}_{11_1}\, r_1r_1r_1 + \underline{g}_{22_2}\, r_2r_2r_2 + \underline{g}_{33_3}\, r_3r_3r_3 + \\
& (\underline{g}_{22_1} + \underline{g}_{12_2})\, r_1r_2r_2 + (\underline{g}_{33_1} + \underline{g}_{13_3})\, r_1r_3r_3 + (\underline{g}_{11_2} + \underline{g}_{12_1})\, r_1r_1r_2 + \\
& (\underline{g}_{33_2} + \underline{g}_{23_3})\, r_2r_3r_3 + (\underline{g}_{11_3} + \underline{g}_{13_1})\, r_1r_1r_3 + (\underline{g}_{22_3} + \underline{g}_{23_2})\, r_2r_2r_3 + \\
& (\underline{g}_{23_1} + \underline{g}_{13_2} + \underline{g}_{12_3})\, r_1r_2r_3
\end{aligned}$$

Since $Q(\underline{r})\, \underline{r}$ must vanish identically, all ten vector coefficients must be zero, leaving only 8 free vector quantities to parameterize Q. The 8 free vector quantities are those in the expression of statement 3).

The constructive formulas for parameterizing $Q(\underline{r})$ and $F(\underline{r})$ give a splitting for the following homological-algebra exact sequences:

Long Exact Sequence:

$$
\begin{array}{cccccccccc}
0 & \longrightarrow & \underline{\lambda}\,\underline{r}^T & \xrightarrow{\text{inclusion}} & F(\underline{r}) & \xrightarrow{\times \underline{\tilde{r}}} & Q(\underline{r}) & \xrightarrow{\frac{\partial^3}{\partial \underline{r}^3} \times \underline{r}} & \mathbf{R}^{30} & \longrightarrow 0 \\
 & & \text{dim } 3 & & \text{dim } 27 & & \text{dim } 54 & & \text{dim } 30 & \\
 & & \underline{\lambda} & & q, \underline{\lambda} & & Q_{ij} & & Q_{ij}/q &
\end{array}
$$

Short Exact Sequence:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & F(\underline{r})/\underline{\lambda}\,\underline{r}^T & \xrightarrow{\times \underline{\tilde{r}}} & Q(\underline{r}) & \xrightarrow{\frac{\partial^3}{\partial \underline{r}^3} \times \underline{r}} & \mathbf{R}^{30} & \longrightarrow 0 \\
 & & \text{dim } 24 & & \text{dim } 54 & & \text{dim } 30 & \\
 & & q & & Q_{ij} & & Q_{ij}/q &
\end{array}
$$

The kernel of each map is exactly the range of the previous map.

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ABSTRACT: Stability of Dynamic Inversion Control Laws Applied to Nonlinear Aircraft Pitch-Axis Models

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Introduction

Dynamic inversion is a nonlinear control technique that has been applied by Honeywell to a variety of realistic aerospace vehicle models with reasonably good results. The list of study applications includes models of the F-14 aircraft, the HARV F-18 aircraft, a McDonnell Douglas model of the NASP vehicle, and a General Dynamics model of a next-generation booster vehicle. The main advantage of dynamic inversion over more conventional linear control techniques is its applicability to the full nonlinear vehicle models.

The theory of dynamic inversion (and nonlinear control in general) is not well understood. Most of what we know about dynamic inversion theory is summarized in the references [E1], [E2], and [MEHH].

This note describes a global stability result for dynamic inversion applied to nonlinear aircraft pitch-axis models. The point is to examine the technique from a mathematical point of view and try to understand why and how it works.

Section 1: Equations of Motion

We concentrate on aircraft pitch-axis models similar to those used in current aerospace vehicle design. The body-axis coordinate system is used. See Figure 1. There are four states:

U = component of velocity in the aircraft longitudinal (x) axis
W = component of velocity in the aircraft vertical (z) axis
Q = vehicle pitch-rate
θ = vehicle pitch attitude relative to local horizontal.

The equations of motion are:

$$\frac{d}{dt}\begin{bmatrix} U \\ W \\ Q \\ \theta \end{bmatrix} = \begin{bmatrix} -WQ \\ UQ \\ 0 \\ Q \end{bmatrix} + \begin{bmatrix} -g\sin(\theta) \\ g\cos(\theta) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{T}{m} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\rho V^2 S \begin{bmatrix} \frac{C_x(\alpha)}{m} \\ \frac{C_z(\alpha)}{m} \\ \frac{cC_M(\alpha)}{I_Y} \\ 0 \end{bmatrix} + \frac{1}{2}\rho V^2 S \begin{bmatrix} \frac{C_{x,\delta}(\alpha)}{m} \\ \frac{C_{z,\delta}(\alpha)}{m} \\ \frac{cC_{M,\delta}(\alpha)}{I_Y} \\ 0 \end{bmatrix} \delta \qquad (1.1)$$

The variables in equation (1.1) have the following meaning:

g = gravitational acceleration (constant),
T = thrust (control input),
m = vehicle mass (constant),
ρ = air density (assumed constant here),
V = speed = $\sqrt{U^2 + W^2}$,
α = angle of attack = $\operatorname{atan}(\frac{W}{U})$,
I_Y = vehicle inertia (constant),

δ = elevator angle (control input),
c = mean aerodynamic chord (constant),
$C_x(\alpha)$, $C_z(\alpha)$, $C_M(\alpha)$ = aerodynamic functions for $\delta = 0$,
$C_{x,\delta}(\alpha)$, $C_{z,\delta}(\alpha)$, $C_{M,\delta}(\alpha)$ = aerodynamic functions due to nonzero δ.

The two control inputs T and δ are assumed to be limited to values within a fixed interval. A reasonable set of ranges for a fighter is $0 \le T \le mg$ and -20 degrees $\le \delta \le 20$ degrees. The aerodynamic functions usually depend on Mach as well as α, we neglect the Mach dependence here. Though this model has a number of special features introduced to simplify analysis, most of the discussion below can be augmented to apply to models of more general type.

Section 2: The Equilibrium Manifold

The model (1.1) has the following form:

$$\dot{x} = f(x,u) \tag{2.1}$$

where x is a vector in R^n and u is a vector in R^m. Let $f_*(x,u)$ denote the Jacobian matrix of the transformation f from $R^n \times R^m$ to R^n. Let U denote the set of allowed control values in R^m. Define the equilibrium manifold

$$\overline{M} = \{(x,u) \mid f(x,u) = 0, \text{ u in U, and } f_*(x,u) \text{ has rank n}\}.$$

Projecting $\overline{M}$ onto the first factor we obtain M, the set of equilibrium states. Note that $\overline{M}$ and M depend on the specified control limits.

We use T and θ as coordinates on M.

Suppose T and θ are fixed. Note that Q = 0, so the only unspecified terms in equation (1.1) are the aerodynamic forces and moment. However, the direct aerodynamic force terms in the U,W degrees of freedom must exactly balance the gravity and thrust forces for an equilibrium to exist. Therefore, the aerodynamic force vector in the U,W tangent space is completely determined, even though we do not know the point (U,W).

We introduce the notion of residualized aerodynamic functions.

Definition 2.1: For the model represented by equation (1.1), the residualized aerodynamic functions are $\overline{C}_x(\alpha)$, $\overline{C}_z(\alpha)$ defined by:

$$\begin{bmatrix} \overline{C}_x(\alpha) \\ \overline{C}_z(\alpha) \end{bmatrix} = \begin{bmatrix} C_x - \dfrac{C_{x,\delta}C_M}{C_{M,\delta}} \\ C_z - \dfrac{C_{z,\delta}C_M}{C_{M,\delta}} \end{bmatrix}(\alpha) \,. \tag{2.2}$$

The meaning of the residualized aerodynamic functions is simple: at any equilibrium state the aerodynamic force vector in the U,W tangent space is:

$$\begin{bmatrix} F_x(U,W) \\ F_z(U,W) \end{bmatrix} = \frac{1}{2}\rho V^2 S \begin{bmatrix} \overline{C}_x(\alpha) \\ \overline{C}_z(\alpha) \end{bmatrix} \tag{2.3}$$

as can be verified by solving the equation $\dot{Q} = 0$ in (1.1). A problem may arise at values of α for which $C_{M,\delta}(\alpha)$ is zero or (relatively) very small in magnitude. We deal with this problem by splitting it into two subproblems:

1) Assume unlimited control authority and work with the full system
2) Restrict attention to a subset of the states on which control authority is adequate

Denote by A the set of α corresponding to equilibrium states: A is all angles from 0 to 2π if $C_{M\delta}(\alpha)$ never vanishes and δ is unlimited. In practice, elevator deflection angle is limited and its control effectiveness degrades at angles of attack too far from zero, so in real applications the set A turns out to be a proper subset.

We define the notion of univalent residualized aerodynamic functions:

Definition 2.2: The residualized aerodynamic functions are called univalent if the function $(\bar{C}_x(\alpha),\bar{C}_z(\alpha))$ defines a one-to-one mapping from A onto its image in the space of directions in the U,W tangent space.

Observation 2.1: Suppose the residualized aerodynamics are univalent. Then for each fixed value of T and θ there is at most one equilibrium state $\bar{x}(T,\theta)$ in M.

The validity of this observation follows easily from the earlier discussion. Once the direction of the aerodynamic force at the equilibrium is known then the value of α is uniquely determined. Then V^2 is determined by the force magnitude, and the equilibrium values of U and W are uniquely determined.

Section 3: Dynamic Inversion

The control problem we consider here is the following:

Statement of the problem: Give an equilibrium state $\bar{x}$, determine a controller $u = K(x)$ so that $\bar{x}$ is a global attractor for the system

$$\dot{x} = f(x,K(x)) \tag{3.1}$$

It is clear that any global attractor for (3.1) must be an equilibrium state. Using dynamic inversion we will address this problem for vehicle models having univalent residualized aerodynamic functions (see Section 2 for definitions).

Dynamic inversion is a very simple technique (see [E1],[E2], and [MEHH]). We assume here that the reader knows the basics.

The approach we take is to invert the rotational degrees of freedom to a set of stable, second order dynamics. We leave the value of T fixed -- we could use T as a control input but we choose not to. The throttle is typically used as a low-bandwidth control that is not changed during dynamic maneuvers. In the rest of this section we show how K is constructed and discuss stability of the attitude dynamics. The big question concerning the stability of the complementary dynamics is addressed in Section 4.

To construct the controller K, first select a desired stable set of second-order linear dynamics for θ:

$$\dot{Q} = -2\,\zeta\omega\,Q - \omega^2\,(\theta - \theta_{cmd}) \tag{3.2}$$

Desirable sets of dynamics of this type for aircraft pitch axis control are described in the MIL-F-8785C specifications document. These dynamics are realized in the closed-loop system if δ takes values according to the following equation:

$$\delta = \frac{\left[\frac{2I_Y}{\rho V^2 Sc}\,\dot{Q}_{cmd} - C_M(\alpha)\right]}{C_{M,\delta}(\alpha)} \qquad (3.3)$$

where $\dot{Q}_{cmd}$ has the value:

$$\dot{Q}_{cmd} = -\,2\,\zeta\omega\,Q - \omega^2\,(\theta - \theta_{cmd}) \qquad (3.4)$$

and θ_{cmd} is the θ value of the desired equilibrium state $\bar{x}$. These equations will do the job so long as they give allowed values of δ.

The surface saturation problem at high α appears in all real aircraft. That is the motivation for using thrust-vectoring for high-α control.

The problem of attitude control by dynamic inversion using thrust vectoring was solved by Mike Elgersma in [E1]. For our current problem, we assume the existence of an invariant set in the closed-loop state space inside which the the pitch dynamics have been stabilized by dynamic inversion using aerodynamic surfaces only.

Recent analysis suggests there is an invariant set closely tied to practical aircraft flight. This set is a bounded region such that:

1) α lies between the zero-lift value $\alpha_0 < 0$ degrees and the maximum lift-to-drag value α_{max},
2) V lies between a minimum cruising speed V_{min} and a maximum speed V_{max},
3) the flight-path angle $\gamma = \theta - \alpha$ lies in a fixed range about zero (say $\pm$ 10 degrees),
4) T is greater than or equal to the value needed to fly maximum γ at the V_{min}/α_{max} condition
5) θ is constrained to lie in an interval about zero consistent with conditions 1 and 3
6) Q is constrained to have magnitude smaller than a suitable ε

Section 4: The Stability Result

For pitch-axis models, computer simulations show that the pitch-attitude control strategy presented in Section 3 has excellent stability properties over extreme ranges of initial and transient state conditions. So far we have applied the technique without a rigorous proof that it should work globally. In this section we give a global stability result that begins to explain the results observed for pitch-axis applications.

We sketch the proofs for the unlimited control authority case.

Lemma 4.1: Suppose the aerodynamic drag is always positive. Then there is a finite disc D centered at the origin in the U,W plane into which all trajectories eventually enter and remain.

Proof: From equation (1.1) we can compute the time rate of change of $V^2 = U^2 + W^2$:

$$\frac{d}{dt}\,(U^2 + W^2) = 2\,gV\sin(\alpha - \theta) + 2\,TV\cos(\alpha) - \rho V^3 S\,\bar{C}_D(\alpha,\delta). \qquad (4.1)$$

The expression $\bar{C}_D$ denotes the total drag coefficient including direct surface effects. The first two terms on the right side are linear in V, while the last is cubic. For sufficiently large V the last term will dominate.

Lemma 4.2: Let $F(U,W) = (F_x(U,W), F_z(U,W))$ denote the residualized aerodynamic force vector defined in equation 2.3

Define the residualized system:

$$\begin{bmatrix}\dot{U}\\ \dot{W}\end{bmatrix} = g\begin{bmatrix}-\sin(\theta_{cmd})\\ \cos(\theta_{cmd})\end{bmatrix} + \begin{bmatrix}\frac{T}{m}\\ 0\end{bmatrix} + \begin{bmatrix}F_x\\ F_z\end{bmatrix} \tag{4.2}$$

Assume

$$\mathrm{div}(F) = \frac{\partial F_x}{\partial U} + \frac{\partial F_z}{\partial W} < 0 \tag{4.3}$$

for all (U,W) not equal to (0,0). Then the only possible closed orbits of the residualized system are equilibria.

Proof: The proof is by contradiction. Let Φ denote the right hand side of equation 4.2. Suppose there is some closed orbit C -- either a limit cycle or a homoclinic orbit. Let A denote the interior of C. By Green's theorem in the plane,

$$\int_C \langle\Phi,\nu\rangle = \int_A \mathrm{div}(\Phi)\, d\mathrm{Area} \tag{4.4}$$

where ν denotes the outward pointing unit normal vector to C. The integrand on the left vanishes identically by construction, while the integrand on the right is equal to div(F) which is negative by assumption. This contradiction proves the lemma.

Remark 4.1: This Lemma (and proof) are given as exercise 1.3 in chapter 2 of [H]. We approach the problem this way because some examples we have looked at (the F-4, F-14, and F-15 aircraft) appear to satisfy the negative divergence condition. See Figures 2 and 3 (copied from [MEHH]).

We have already observed in Section 2 (Observation 2.1) that for models having univalent residualized aerodynamics there is a unique equilibrium state when T and θ are fixed. Under the additional condition that the residualized aerodynamic vector field has negative divergence, it follows by the Poincaire-Bendixson theory [H] that when T and θ are held fixed and the aerodynamics have negative divergence, the unique equilibrium state is a global attractor for the U,W dynamics inside the invariant set.

Section 5: Linear Models

In the linear case the issue of stability is related to right half plane zeros in the elevator to pitch angle transfer function. Under the usual assumptions [McAG] this transfer function is given by

$$\frac{\theta}{\delta} = \frac{A(s+\frac{1}{T_1})(s+\frac{1}{T_2})}{(s^2+a_{ph}s+b_{ph})(s^2+a_{sp}s+b_{sp})} \tag{5.1}$$

where both zeros are typically in the left half plane with a few exceptions: there are a few conditions where $\frac{1}{T_1}$ is negative and there are post stall conditions where $\frac{1}{T_2}$ is negative.

The application of dynamic inversion to the pitch axis in the linear case gives a closed loop characteristic equation given by

$$\phi_{cl}(s) = (s^2+2\zeta\omega s+\omega^2)(s+\frac{1}{T_1})(s+\frac{1}{T_2}) \tag{5.2}$$

so right half plane zeros of $\frac{\theta}{\delta}$ give unstable closed loop poles but yet

$$\frac{\theta}{\theta_c}=\frac{\omega^2}{(s^2+2\zeta\omega s+\omega^2)} \tag{5.3}$$

where ω and ζ are specified. We conclude:

Observation 5.1: the dynamic inversion controller discussed in Section 3 for the linear model is stable if and only if T_1 and T_2 are positive.

Observation 5.1 makes it easy to check the stability of the closed-loop system by looking at standard tabular data for the open-loop zeros. The open-loop parameters for the transfer functions of 10 aircraft at 94 total flight conditions is tabulated in Heffley and Jewell's data book [HJ]. Of these 94 flight conditions there were four with right half plane zeros: the four exceptional values were in the range $-0.027 \le \frac{1}{T_1} \le -0.00049$ rad/sec. None of the flight conditions were post stall so $\frac{1}{T_2}$ was positive for all 94 conditions.

These are low frequency instabilities that pilots or autopilots usually stabilize with a combination of additional outer loop feedbacks: the outer-loop feedback maps a combination of altitude, altitude rate, and velocity to a combination of elevator and throttle.

In [McAG] there are approximate formulas for $\frac{1}{T_1}$ that show that it is the variation of thrust minus drag with velocity that causes the right half plane zeros for 3 of the 4 exceptional flight conditions. Since drag increases with velocity (stabilizing) we conclude the right half plane zeros result in part from propulsion characteristics (consequently, the stability result of Section 4 should not be expected to apply). These 3 cases were F-4 at Mach 0.6 and 35K ft altitude, and the XB-70 at Mach 0.6 and 20K ft and Mach 0.9 and 40K ft altitude.

The one other exceptional case is interesting. That case is the X-15 at Mach 1.6 and 80K ft altitude. The right half plane zero is not associated with propulsion because the engine was off. We do not know why the stability result of Section 4 does not work here (other than to say that some hypothesis was violated). This flight condition was exceptional in that the angle of attack above 14 degrees was the largest of all those listed for the X-15 (10 total).

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Figure 1

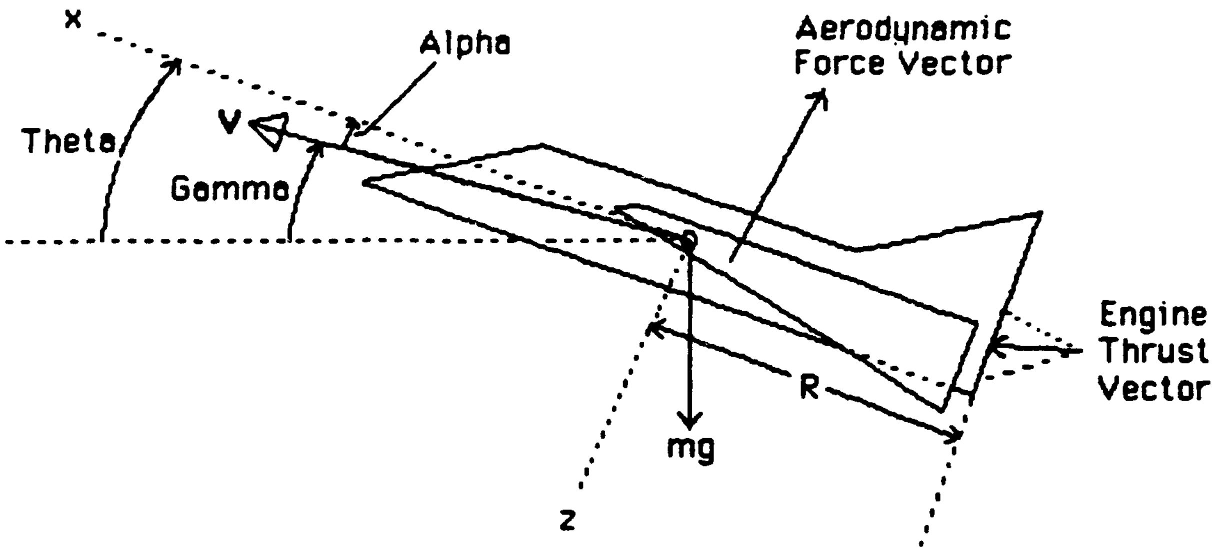
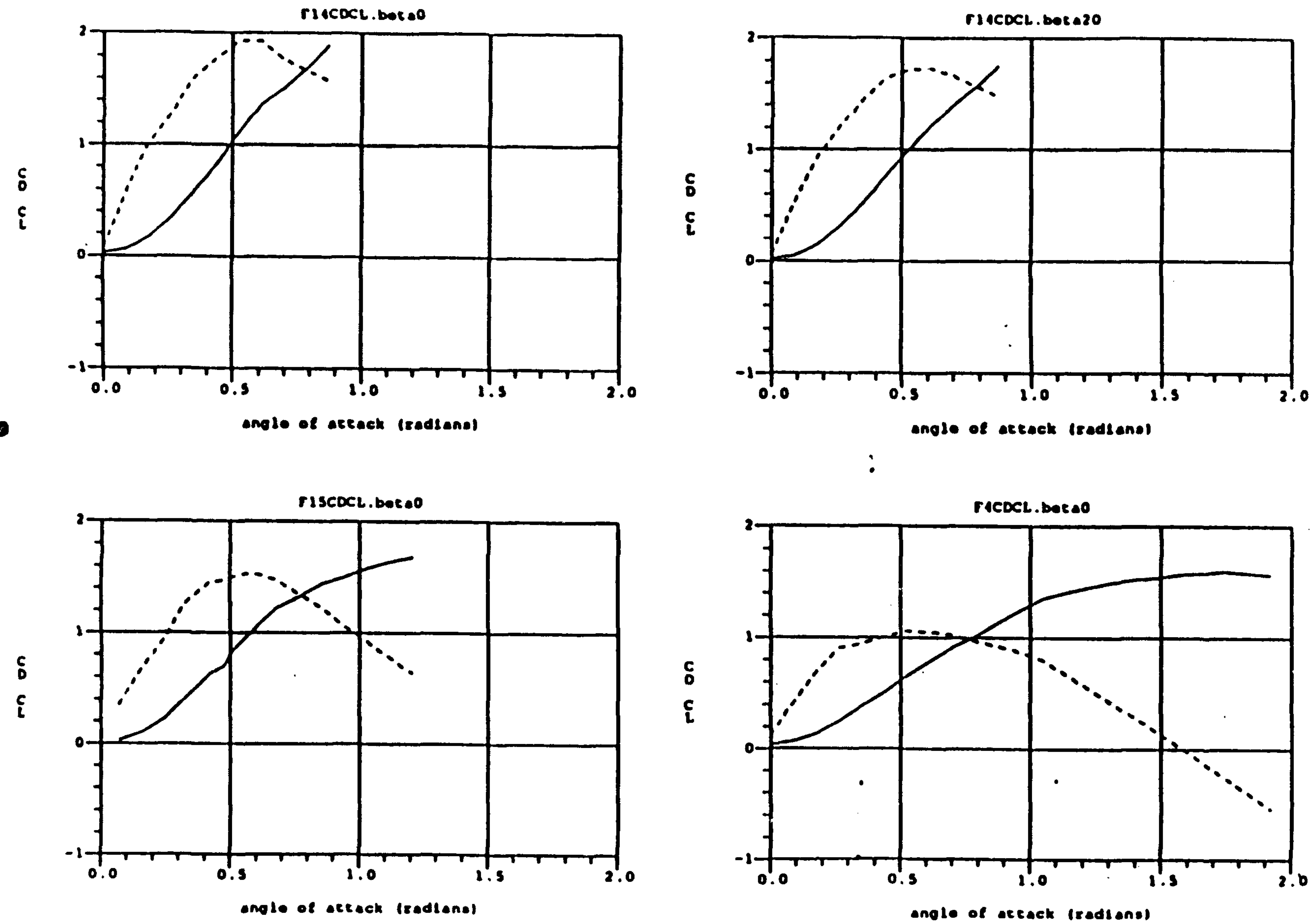
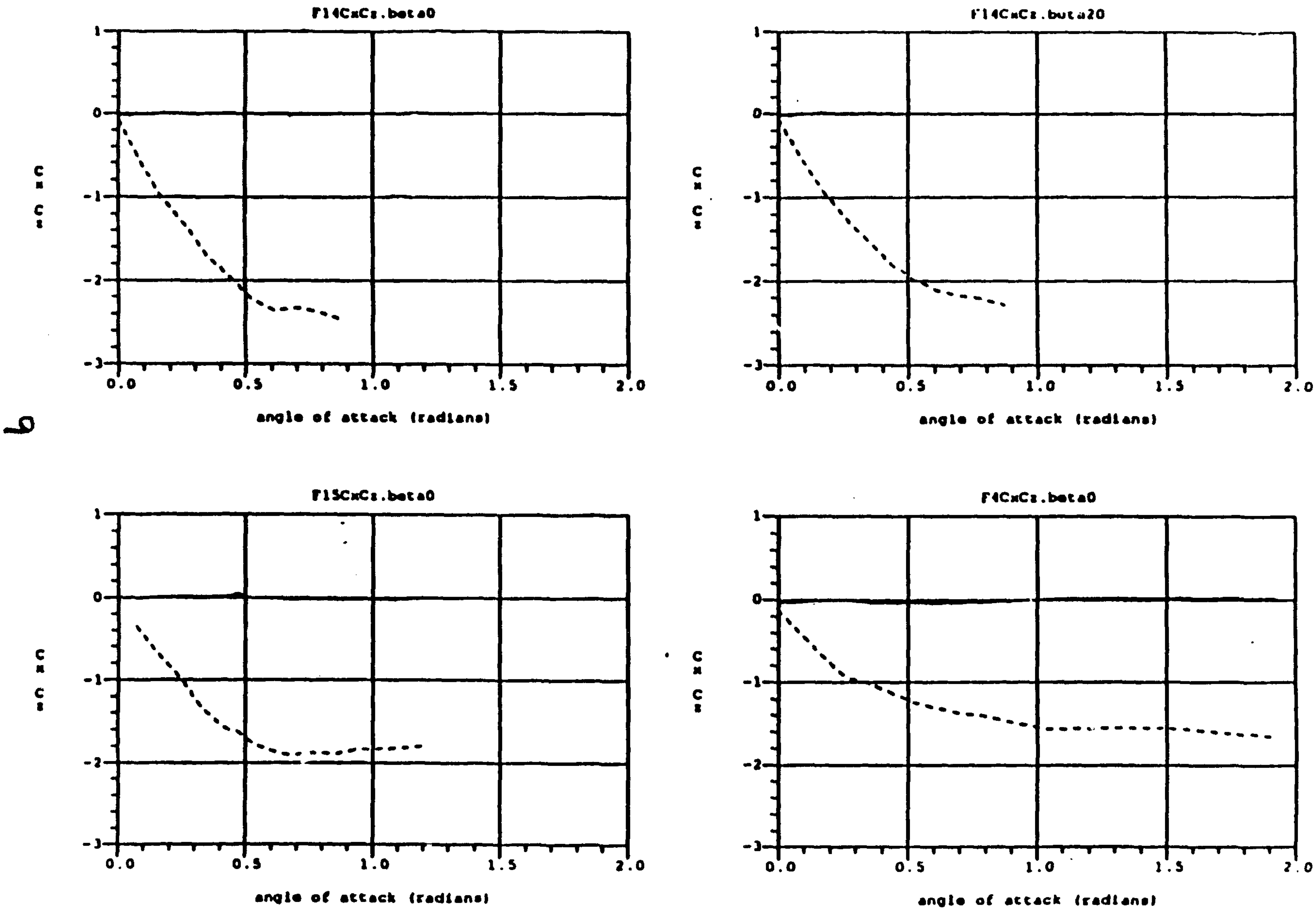

x
Alpha
Aerodynamic
Force Vector
Theta
V
Gamma
Engine
Thrust
Vector
R
mg
z

Figure 2



Drag and Lift Data for the F4, F14, and F15

Figure 3



Axial and Normal Force Data for the F4, F14, and F15